THE

MATHEMATICAL GAZETTE

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CHANGE OF ADDRESS.

THE Editor will shortly be changing his address, when communications should be addressed to him, *by name*, as follows :

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As the Editor will be on leave for part of August, he would be grateful if communications which are not urgent were not sent to him till September.

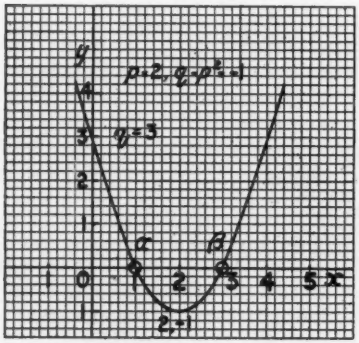
NOTICE TO "MAIN SCHOOL" TEACHERS.

ALL teachers who are concerned with the syllabus of the subject "Elementary Mathematics" in the School Certificate Examinations should see the new alternative syllabus of Geometry and Trigonometry in the examination of the Cambridge Local Syndicate to which reference was made in a letter from the Secretary of the Syndicate published in the May *Gazette*. Copies of the syllabus may be had on application to the Local Examinations Syndicate, Syndicate Buildings, Cambridge. Although the new syllabus is optional and experimental, it is important that the schedules of theorems and constructions and the specimen paper included in the circular should be studied and discussed by members of the Association. For it is likely that we shall be called upon to give a considered judgment on the new scheme and on other changes in the syllabus of "Elementary Mathematics" which may be desirable.

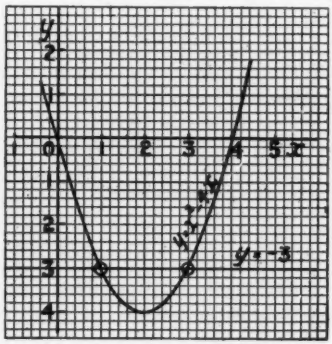
THE GRAPHICAL TREATMENT OF QUADRATIC EQUATIONS.

By C. W. Hansel.

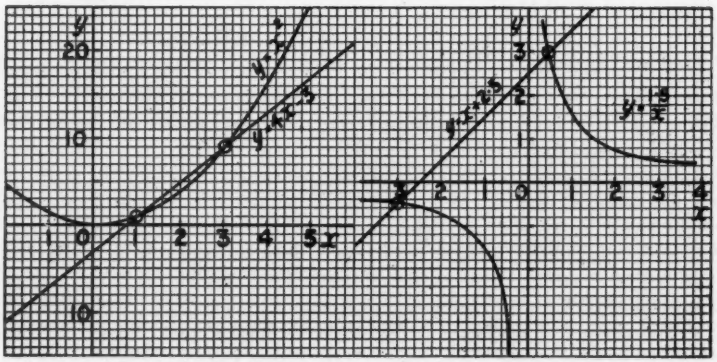
The object of the present article is to indicate the scope of graphical methods and the desirability of using them far more extensively in teaching elementary mathematics. Articles indicating some of the possibilities have appeared recently in the *Mathematical Gazette*, one by Dr. W. G. Bickley, "Arithmetic, Algebra, and Calculus on the Drawing Board", May 1941, Vol. XXV, No. 264, and another by C. Dudley Langford, "A graphical method of solving problems on 'Rate of Work' and similar problems", Dec. 1941, Vol. XXV, No. 267. Graphical work is so universal in its application that it is impossible to deal with more than a very small section of work in the compass of a single article, and "Quadratic Equations" has been chosen for purposes of illustration. Through the medium of graphical methods and concrete numerical illustrations, the elementary student of mathematics gets a much clearer and more interesting view of the subject.



1.1. $x^2 - 4x + 3 = 0$.



1.2. $x^2 - 4x + 3 = 0$.



1.3. $x^2 - 4x + 3 = 0$.

1.4. $2x^2 + 5x - 3 = 0$.

I. Methods of Solution.

1. *Algebraic Methods.*

Let α and β be the roots of $x^2 - 2px + q = 0$.

$$\alpha + \beta = 2p \; ; \; \alpha\beta = q.$$

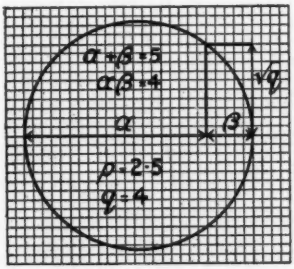
The quadratic function $y = x^2 - 2px + q$ has a minimum value $y_m = q - p^2$ when $x = p$.

The cartesian plot of y against x is a parabola which cuts the axis of x at α and β, the axis of y at q, and the coordinates of the minimum point are $(p,\ q - p^2)$.

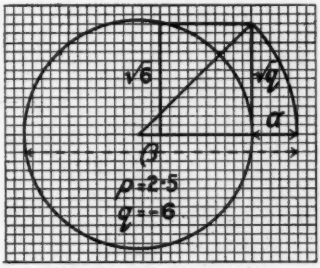
1.1 exhibits the properties of the function $y = x^2 - 2px + q$.

1.2, 1.3, 1.4 indicate simple graphical methods of solving quadratic equations which become laborious when p and q have values such as would be appropriate to most practical problems.

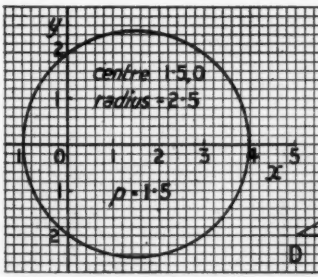
2. *Geometrical Methods.*



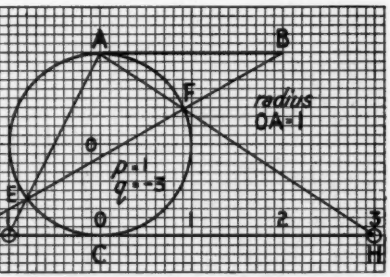
2.1. $x^2 - 5x + 4 = 0$.



2.2. $x^2 - 5x - 6 = 0$.



2.3. $x^2 - 3x - 4 = 0$;
$y^2 + x^2 - 3x - 4 = 0$.



2.4. $x^2 - 2x - 3 = 0$;
$AB = 2/p = 2$, $CD = q/2p = 3/2$.

2.1 and 2.2 show well-known simple geometrical constructions.

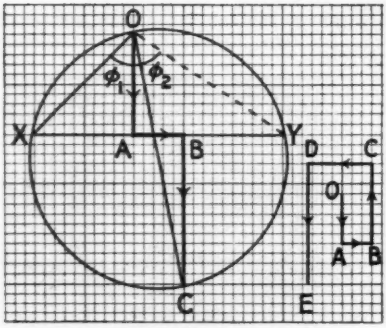
In 2.3, the centre and radius of the circle are found from the equation of the circle $x^2 + 2gx + c + y^2 = 0$.

2.4 is a geometrical construction that can be developed into a nomogram.

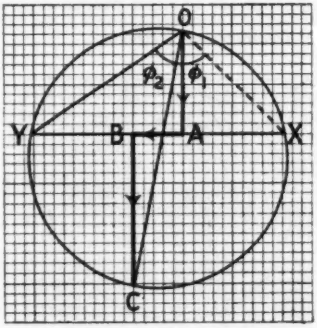
2.4 is given in Hardy's *Pure Mathematics*, p. 8, where reference is made to Klein's *Leçons sur certaines questions de Géométrie Elémentaire* (French translation by J. Greiss, Paris, 1896).

The construction is also given in Klein's *Famous Problems of Elementary Geometry*, p. 34, in the discussion of how to construct a regular polygon of 17 sides with ruler and compasses only. It is there stated: "The construction of the regular polygon of 17 sides is made in accordance with the methods indicated by Poncelet and Steiner, inasmuch as besides the straight edge but one fixed circle is used. First, we will show how with the straight edge and one fixed circle we can solve every quadratic equation."

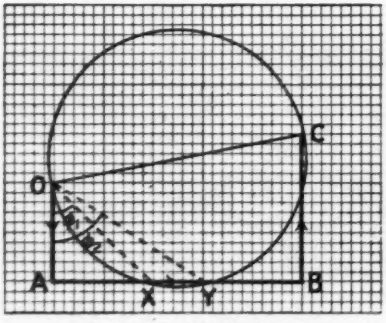
He then uses the construction to solve four quadratic equations which enable the polygon of 17 sides to be constructed (Fig. 9, p. 41).



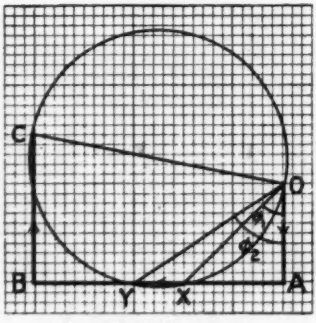
3.1. $2x^2+x-3=0$;
$\alpha=\tan\phi_1=1$, $\beta=\tan\phi_2=-1{\cdot}5$.



3.2. $2x^2-x-3=0$;
$\alpha=\tan\phi_1=-1$, $\beta=\tan\phi_2=1{\cdot}5$.



3.3. $2x^2+5x+3=0$;
$\alpha=\tan\phi_1=-1$, $\beta=\tan\phi_2=-1{\cdot}5$.

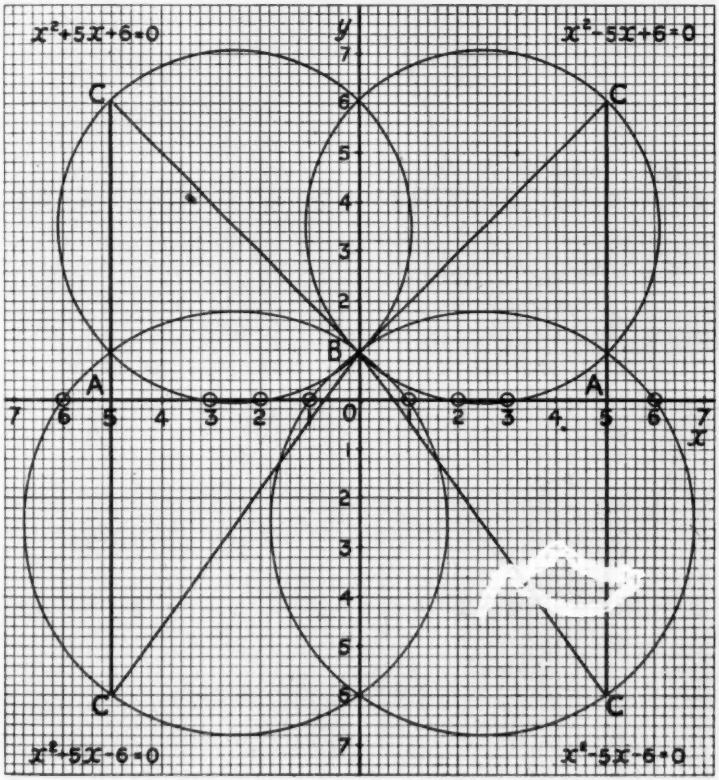


3.4. $2x^2-5x+3=0$;
$\alpha=\tan\phi_1=1$, $\beta=\tan\phi_2=1{\cdot}5$.

3.1 to 3.4 are explained in Dr. Bickley's article already mentioned. See also *Calcul Graphique et Nomographie* by D'Ocagne and *Graphical Methods*

by Carl Runge. To solve $ax^2+bx+c=0$, draw $OA=a$, AB at right angles to OA and $=b$, BC at right angles to AB and $=c$. Draw a circle on OC as diameter cutting AB or AB produced in X and Y. If angle $XOA=\phi_1$ and angle $YOA=\phi_2$, $\alpha=\tan\phi_1$, $\beta=\tan\phi_2$.

The small diagram 3.1 gives the positive directions for the lines OA, OB, OC, etc. The opposite direction is negative. If the equation is reduced to the form $x^2-2px+q=0$, $OA=1$, $AB=-2p$, $BC=q$. The roots of the equation are now XA and YA. This method does not give the imaginary roots if $b^2<4ac$.



4.1. 4.2.
4.3. 4.4.

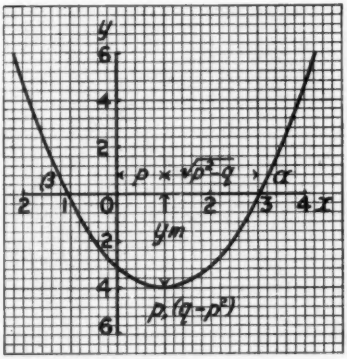
Thomas Carlyle's method is illustrated in 4.1 to 4.4 (for real roots) and 6.1 and 6.2 (for imaginary roots). The construction is carried out as follows: Using rectangular axes OX, OY, B is the point 0, 1; A is $2p$, 0; C is $2p$, q. A circle is drawn on BC as diameter. If this circle cuts OX, the roots are real and equal to the x-coordinate of the points of cutting.

3. *Imaginary Roots.*

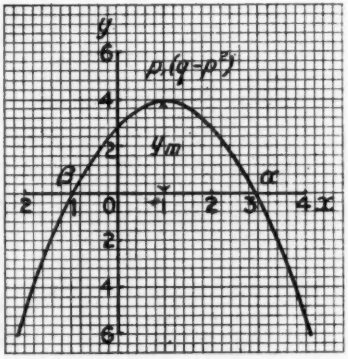
If $q > p^2$ the roots of the quadratic $x^2 - 2px + q = 0$ are imaginary and equal to $p \pm j\sqrt{y_m}$ where y_m is the minimum value of the function $x^2 - 2px + q$, which is $q - p^2$ when $x = p$.

5.1 to 5.4 indicate various types of solution using the properties of the minimum or maximum point.

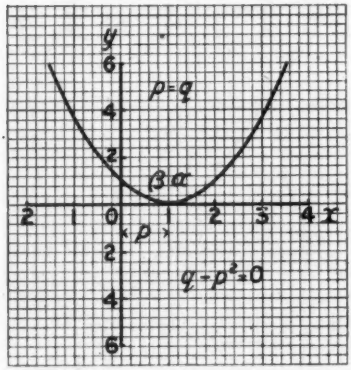
6.1 and 6.2 illustrate Carlyle's method when the tangent to the circle t is used to give the value of $(p^2 - q)$; $t^2 = p^2 - q$.



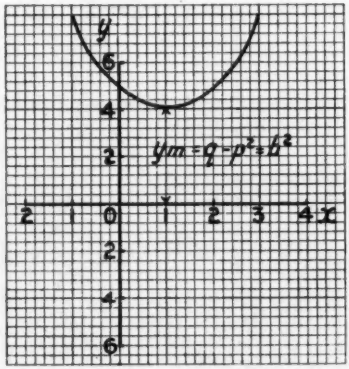
5.1. $y = x^2 - 2x - 3$.



5.2. $y = -(x^2 - 2x - 3)$.



5.3. $y = x^2 - 2x + 1$.



5.4. $y = x^2 - 2x + 5$;
$\alpha = a + jb = p + j\sqrt{y_m}$,
$\beta = a - jb = p - j\sqrt{y_m}$.

4. *General Methods of Solution.*

General methods of solution usually involve non-uniform scales or families of curves.

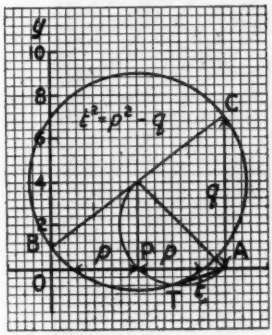
6.3 is a modification of the diagram given in D'Ocagne's *Calcul Graphique et Nomographie* and attributed to M. Gercevanoff. The discussion given in

Calcul Graphique relates only to negative values of q. 6.3 shows a construction for positive values of q as well as the construction of M. Gercevanoff.

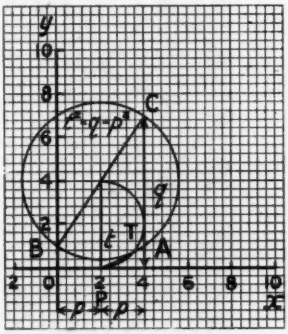
Consider rectangular axes Ox, Oy. Ox is a uniformly divided scale, any scale number being either positive or negative. Oy supports a "square root" scale, the unit distance being the same as for Ox. The scale numbers along Oy are at distances from O equal to the square root of the scale number. The numbers represent $\pm q$.

To solve $x^2 - 5x - 6 = 0$.

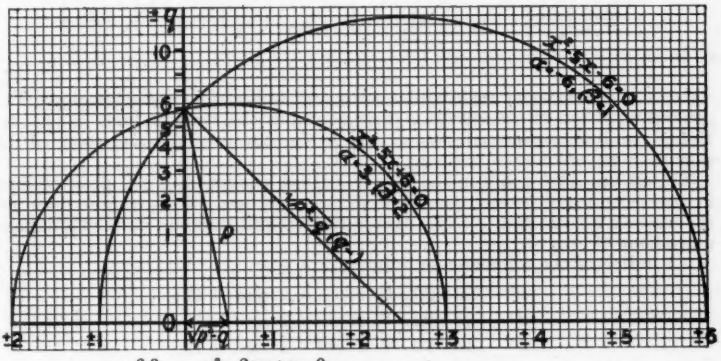
$q = -6$; $p = 2{\cdot}5$; $\sqrt{(p^2 - q)}$ is obtained by joining $q = -6$ to $x = 2{\cdot}5$. This join is of length $\sqrt{(p^2 - q)}$. A circle with centre 2·5, 0 and radius $\sqrt{(p^2 - q)}$ cuts Ox at ± 6 and ± 1. $\alpha = -6$; $\beta = +1$.



6.1. $x^2 - 8x + 7 = 0$.



6.2. $x^2 - 4x + 7 = 0$;
$\alpha = p + j\sqrt{y_m} = 2 + j\sqrt{3}$,
$\beta = p - j\sqrt{y_m} = 2 - j\sqrt{3}$.

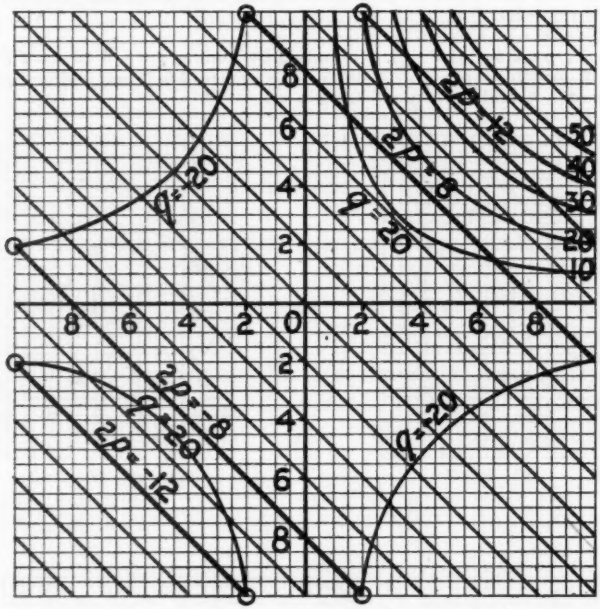


6.3. $x^2 - 2px + q = 0$.

If q is positive, the numbers to the right and left of O are of the same sign as p. If q is negative (as in Gercevanoff's construction), the positive numbers are to the left of O and the negative numbers to the right of it.

To solve $x^2 - 5x + 6 = 0$.

In this case, $q = +6$; $p = 2{\cdot}5$; $\sqrt{(p^2 - q)}$ is obtained along Ox, the distance p being set off from the point O, q as shown in 6.3 (the join of 0, 6 and $\sqrt{(p^2 - q)}$, 0). A circle with centre $\sqrt{(p^2 - q)}$, 0 and radius p, cuts Ox at ± 3 and ± 2. The roots are $\alpha = +3$; $\beta = +2$.



7.1.

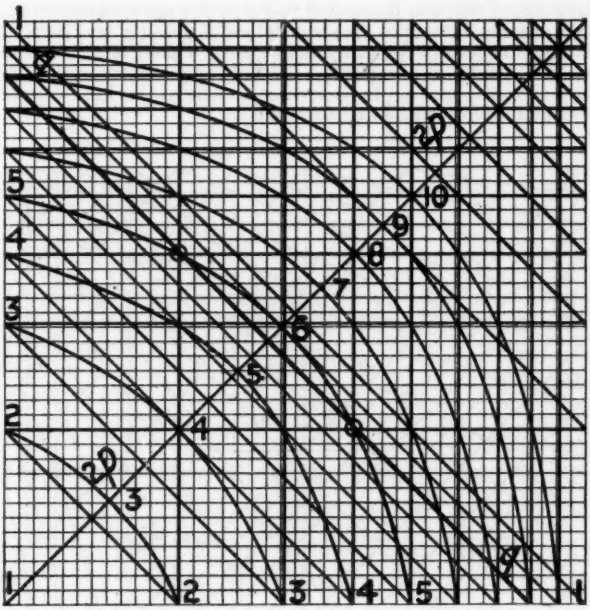
7.1 is a chart using rectangular axes and uniformly divided scales. The general equation of the curves is $xy = q$ and of the straight lines $x + y = 2p$. Hence the intersections of $xy = q$ and $x + y = 2p$ give the real roots of the quadratic equation $x^2 - 2px + q = 0$.

The curves $xy = q$ are in the first and third quadrants if q is positive, and in the second and fourth quadrants if q is negative. The straight lines cut the axes at $x = 2p$ and $y = 2p$. Hence the straight lines are parallel and have a gradient of -1.

7.1 indicates the following solutions :

$$x^2 - 8x - 20 = 0, \text{ roots } -2 \text{ and } +10 ;$$
$$x^2 + 8x - 20 = 0, \text{ roots } +2 \text{ and } -10 ;$$
$$x^2 - 12x + 20 = 0, \text{ roots } +2 \text{ and } +10 ;$$
$$x^2 + 12x + 20 = 0, \text{ roots } -2 \text{ and } -10.$$

7.2 is a chart using rectangular axes Ox, Oy and non-uniform (logarithmic) scales. The straight lines $xy = q$ (q positive) have intercepts q on each of the axes. The curves $x + y = |2p|$ are easily drawn.



7.2.

7.2 indicates that the roots of $x^2 - 6x + 8 = 0$ are $+2$ and $+4$.
The roots of the equation $x^2 + 2px + q = 0$ are equal to those of

$$x^2 - 2px + q = 0$$

but of opposite sign. Changing the sign of p does not alter the value of the roots but changes the sign of both of them.

7.3 is a chart of logarithmic scales. The curves are in pairs: $x - y = 2p$ and $y - x = 2p$. The equation of the straight lines is $xy = |q|$.

7.3 indicates that the roots of $x^2 - 2x - 8 = 0$ are $+4$ and -2.

7.4 is a chart plotted to rectangular axes and with uniformly divided scales.

The equation $x^2 - 2px + q = 0$ may be written in the form $q = 2x \,.\, p - x^2$. Treating x as a parameter, and assigning a succession of values to x, a family of straight lines is obtained having gradient $2x$ and intersecting the axis of q at $0, -x^2$.

If $x = \pm 1$, $q = \pm 2x - 1$;

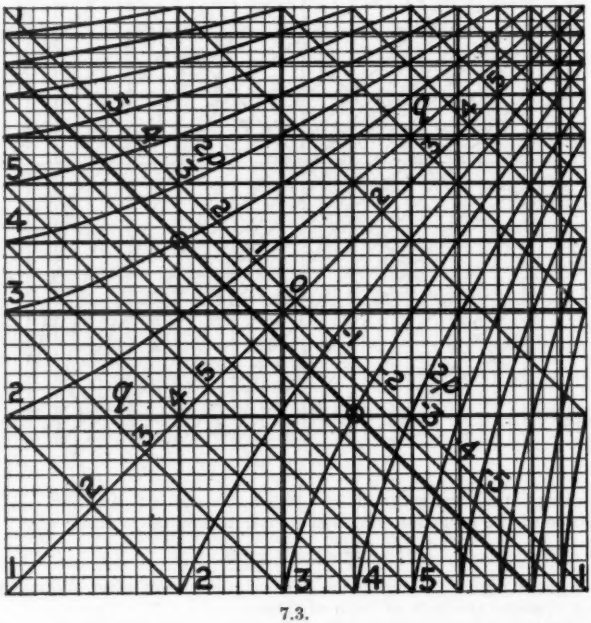
if $x = \pm 2$, $q = \pm 4x - 4$;

if $x = \pm 3$, $q = \pm 6x - 9$;

if $x = \pm \frac{1}{2}$, $q = \pm x - \frac{1}{4}$; etc.

7.4 shows this family of straight lines and indicates that the roots of $x^2 - 2x - 15 = 0$ are $+3$ and -5, since the lines $x = 3$ and $x = -5$ intersect at $2p = 2$ and $q = -15$.

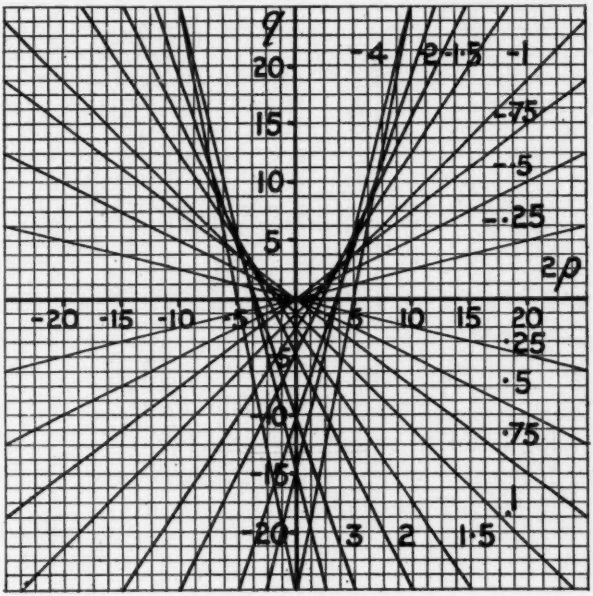
The methods of solution illustrated in 7.1 to 7.4 are unsatisfactory, partly on account of scale difficulties for some values of p and q, and partly because interpolation is necessary for many non-integral values of x. In some practical problems, values of a variable x are only required for certain values of parameters such as p and q. In such cases a chart of this kind may be useful and convenient, giving all desired values of x.



7.3.

5. *Alignment Charts or Nomograms or Abacs.*

M. Phillippe Tongas in *Mécanique*, Mars-Avril 1938, pp. 59–69, distinguishes between two types of chart. Those depending on the concurrency of lines he calls " abaques " and those depending on collinearity of points " nomogrammes ". The term " nomogramme " is due to D'Ocagne, who as early as 1899 gave such a complete exposition of the subject of Nomography in his *Traité de Nomographie* that little has since been added except matters of detail. The term " Nomogram " is therefore entitled to considerable respect, but it has been criticised as a misnomer, as it connotes the writing of laws although the word also implies an orderly arrangement. The term " abac " is short and suggests " abacus ", the universal instrument of computation in ancient times. The Greek $A\beta\alpha\xi$ was also used for teaching the principles of writing and geometry. The abax was strewn with green sand and a small rod or radius was used to trace letters or geometrical figures in the sand. (See *Ency. Brit.* VIIIth Ed., article " Abacus " by Sir John Leslie, and Sir John Leslie's *Arithmetic*.)



7.4.

8.1 is D'Ocagne's abac for the positive roots of $x^2+px+q=0$ (see *Traité de Nomographie* or *Calcul Graphique et Nomographie*). A straight line connecting the values of p and q cuts the curved scale z at scale numbers equal to the positive roots of the equation.

In what follows, the following notation will be used :

The scales of an abac are indicated by a number or letter enclosed by a square bracket.* The figure or scale number or graduation number is placed in front of the bracket. Thus " 3 [x] " means " the scale number 3 on the x scale ", and may be read " 3 on x ".

" 3 [x] $\equiv$ 9 [x^2] " means " 3 on scale x is opposite 9 on scale x^2 ".

The scale numbers 3 and 9 are " corresponding graduation numbers " and could either be together on adjacent scales or on the same horizontal or vertical line on separated scales.

" 3 [1] : 4 [2] $\rightarrow$ 5 [3] " means " the join of 3 on 1 and 4 on 2 passes through 5 on 3 ".

A join functioning as an alignment device is a " secor ".

In German this is the " ablesegerade " or the " reading off line ".

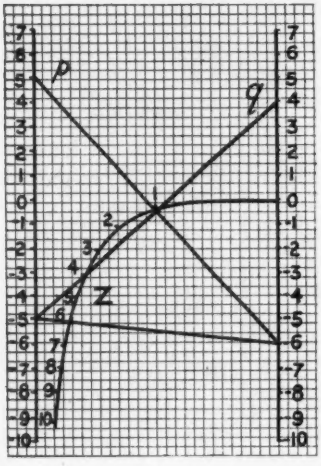
To solve $x^2-5x+4=0$, using 8.1,

$$-5\,[p] : 4\,[q] \rightarrow 1 \text{ and } 4\,[z].$$

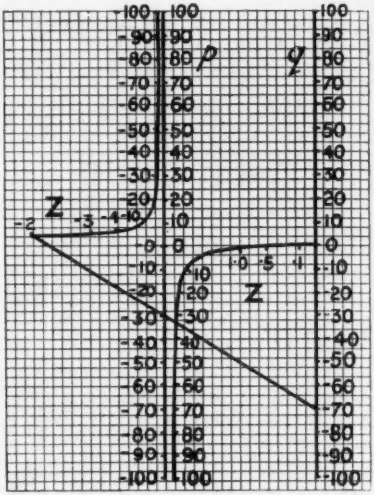
To solve $x^2-5x-6=0$, using 8.1,

$$-5\,[p] : -6\,[q] \rightarrow 6\,[z], \textit{ i.e.} \text{ the positive root.}$$

* In diagrams 9–11 the scales are indicated by " (x) ".

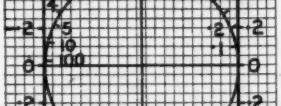


8.1. $z^2+pz+q=0$;
D'Ocagne, *Calcul Graphique*, p. 270.



8.2. $z^2+pz+q=0$;
Hewes and Seward, *Design of Diagrams*, p. 51.

8.3. $x^2+px-q=0$;
Van Voorhis, *Alignment Charts*, p. 31.



8.4. $cu^2+u+d=0$;
Whittaker's nomogram.
Van Voorhis, Fig. 58.

The negative root is found by changing the sign of the positive root of $x^2+5x-6=0$, since changing the sign of p changes the sign but not the value of roots.

$5\,[p]:-6\,[q]\rightarrow+1\,[z]$. Thus the negative root of $x^2-5x-6=0$ is -1.

8.2 shows the form of abac given in *The Design of Diagrams for Engineering Formulae* by Hewes and Seward, p. 51. In this diagram the negative branch of the curve z is drawn so that positive and negative roots are determined by the intersections of a secor connecting p and q and the two branches of the curve. For example, the roots of $x^2-x-6=0$ are collinear with $p=-1$ and $q=-6$, but the scale is inconvenient as it is for the solution of $x^2-5x-6=0$.

8.3 is a modified D'Ocagne abac as given in *How to Make Alignment Charts* by Van Voorhis, p. 81. Two curves are drawn for ranges of values of p from 4 to -11 and q from 4 to -11 and from 40 to -110. In addition, the horizontal scale for p/q permits the use of values of p or q above 4.

If the p-scale numbers are multiplied by a scale factor of 10^n (n a positive or negative integer), the scale factor for the x-scales is 10^n, for the p/q-scale it is 10^{-n}, and for the q-scale it is 10^{2n}.

8.4 shows E. T. Whittaker's abac (see *Dictionary of Physics*, article on "Nomography", Vol. III, p. 635, by S. Brodetsky, and *How to Make Alignment Charts* by Van Voorhis, p. 83). The roots of the quadratic equation $cx^2+x+d=0$ are at the intersections of a circle and a secor connecting c and d.

Whittaker's nomogram (*Edin. Math. Soc. Notes*, No. 15, Dec. 1915, p. 215) is given in the form illustrated in 12.1 (p. 122) in the *Calculus of Observations* by Whittaker and Robinson, p. 129.

8.1 to 8.4 do not give the imaginary roots.

9.1 to 9.4 indicate how to develop an abac to give the real and the imaginary values of the roots of a quadratic equation $x^2-2px+q=0$ in the form $a\pm jb$ for a wide range of values of p and q.

9.1 is the well-known abac giving $c^2=a^2+b^2$ or $q=p^2+b^2$.

$$a\,[1]:b\,[5]\rightarrow c\,[3];\quad p\,[1]:b\,[5]\rightarrow q\,[4].$$

The roots of $x^2-2px+q=0$ are $a\pm jb$ where $a=p$ and $b=\sqrt{(q-p^2)}$;

$$p\,[1]:q\,[4]\rightarrow b\,[5].$$

Example: $x^2-16x+100=0$.

$$a=p=8;\quad q=100;\quad 8\,[1]:100\,[4]\rightarrow 6\,[5].$$

Hence the roots are $8\pm j.6$.

9.1 gives variable accuracy owing to the closeness of [1] and [5] near the zero values. The ranges of values of p, q, and b are also restricted.

9.2 indicates how various ranges of values of p, q, and b may be used and the accuracy of the method improved.

In 9.2, $q=p^2+b^2$ using corresponding scales, *e.g.* 1, 3, 5 or 2, 4, 6. More scales may be added, and scale factors 2, $\frac{1}{2}$, 10, or 1/10 may be applied to the scales for p and b.

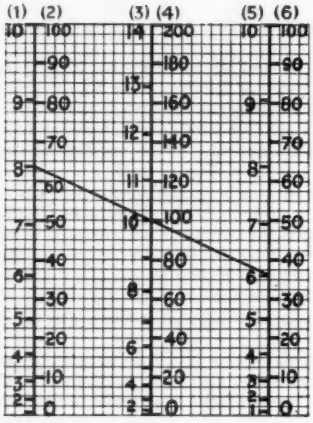
It will be noticed that a scale factor k applied to p or b gives a scale factor k^2 for q, since $(kp)^2+(kb)^2=k^2q$.

9.3 indicates how the range of values of p, q, b can be still further increased.

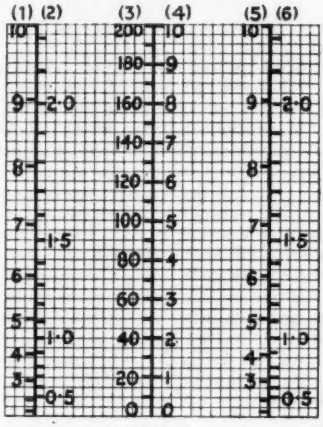
If $x^2-2px+q=0$, $x=a\pm j.b$ where $a=p$ and $b=\sqrt{(q-p^2)}$. To find b:

$$p\,[p_m]:b\,[b_n]\rightarrow q\,[q_{m+n}].$$

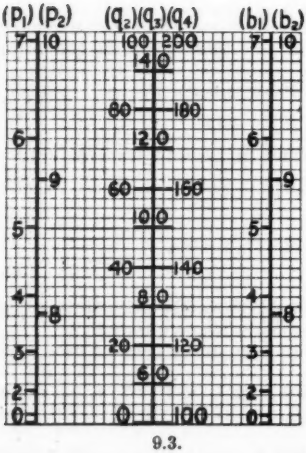
9.4 is an abac for solving any quadratic equation except for values of p and q which differ considerably (in which case the method of approximation



9.1.



9.2.



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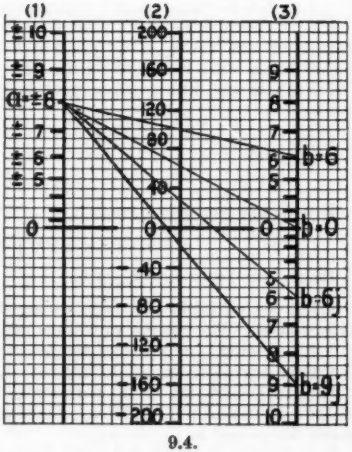


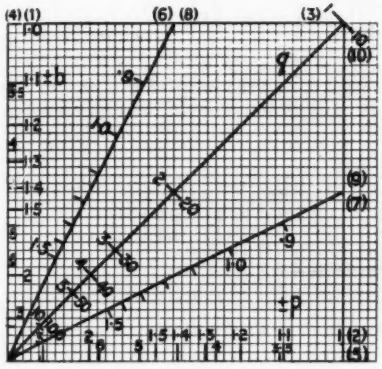
9.4.

given later is used). In this abac, positive and negative values of p and b occur together on the positive part of the scale. The negative part of these scales represents a rotation of the positive part through 180° and for the uniform scales multiplication by j^2, but for the square scales this becomes j. Hence imaginary numbers are represented below the zero on the scales for p and b. Negative values of q occur on this uniform scale below the zero.

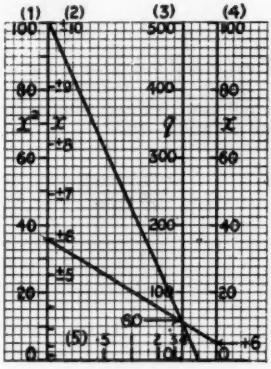
Any quadratic equation may be solved by drawing a secor connecting the values of p and q. For example :

$x^2 \pm 16x + 100 = 0$; roots $8 \pm 6j$ or $-8 \pm 6j$.
$x^2 \pm 16x + 64 = 0$; roots 8 or -8.
$x^2 \pm 16x + 28 = 0$; roots $14, 2$ or $-14, -2$.
$x^2 \pm 16x - 17 = 0$; roots $17, -1$ or $-17, 1$.

Multiple scales may be used as in 9.3.



10.1.



10.2.

10.1 is an abac for $q = p^2 + b^2$ using reciprocal scales (Von O. Heck und A. Walther, *Ingenieur Archiv.* I, 1930). [1], [4], [6], and [8] represent the reciprocal of $1/b^2$; [2], [5], [7], [9] represent the reciprocal of $1/p^2$; and [3], [10] represent the reciprocal of $1/q$.

$p\,[2] : q\,[3] \rightarrow b\,[1]$; $p\,[5] : q\,[10] \rightarrow b\,[4]$.
$p\,[7] : q\,[3] \rightarrow b\,[6]$; other scales may be added.

This abac may sometimes be preferred to 9.4.

10.2 is an abac with a movable scale, [3], which may either be drawn in a suitable position or it may be drawn on transparent paper and placed on a drawing of [1] and [4]. A horizontal segmentary scale, [5], gives the position of [3] at the value $-2p$ on [5]. Fig. 10.2 solves $x^2 + 4x = 60$.

6. *Slide Rule Methods.*

The following notation is convenient. Letter the scales A, B, C, D as usual. a, b, c, d denote graduation numbers on A, B, C, D (whether figured or not). In the ordinary type of rule, $a\,[A] \equiv d\,[D]$ and $a = d^2$.

If the slide is moved so that $a\,[A] \equiv b\,[B]$, this operation may be indicated by $b\,[B] <\!\!=\!\!=\!\!> a\,[A]$, which reads " scale B is moved until the graduation number b on it is opposite graduation number a on the fixed scale A ". The scale letter for the movable scale precedes the sign of operation and the scale letter for the fixed scale follows it.

11.1 illustrates the solution of $x^2 + 2x - 8 = 0$ or $x(x+2) = 8$. In this case the roots are of opposite sign and differ by 2 in absolute value.

$$\left.\begin{array}{l} c\,[C] <\!\!=\!\!=\!\!> 8\,[D] \\ 1\,[C] <\!\!=\!\!=\!\!> d\,[D] \end{array}\right\} \text{ where } c - d = 2,$$

i.e. move the slide until the scale number $c\,[C] \equiv 8\,[D]$ is 2 more than the scale number $d\,[D] \equiv 1\,[C]$. It will now be found that $1\,[C] \equiv 2\,[D]$. Hence the larger root is -4 and the smaller root is 2, the product of the roots being -8 and the difference in absolute value is 2.

The solution of $x^2 + 3x - 28 = 0$ may be carried out in a similar way using $[A]$ and $[B]$.

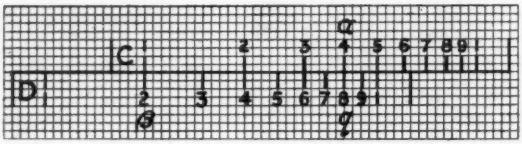
To solve $x^2 - 11x + 24 = 0$ or $x(x-11) = -24$, q being positive.

Set the cursor line over 24 $[A]$. Move the slide until

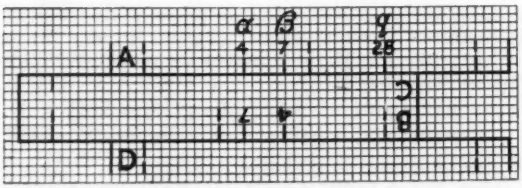
$$\left.\begin{array}{l} b\,[B] \leftrightarrows 24\,[A] \\ 1\,[B] \leftrightarrows a\,[A] \end{array}\right\} \text{ give } a+b=11.$$

Then a and b are the roots of the quadratic (3 and 8).

If $q > 100$, for example $x^2 - 20x - 1500 = 0$, find the roots of $x^2 - 2x - 15 = 0$ which are 10 times smaller (5 and -3). Hence the roots are 50 and -30.



11.1. $x^2 + 2x - 8 = 0$; $x(x+2) = 8$; $\alpha \sim \beta = 2$.



11.2. $x^2 + 3x - 28 = 0$; $x(x+3) = 28$; $\alpha \sim \beta = 3$.

11.2 gives the solution of $x^2 + 3x = 28$.

Remove the slide and replace it in an inverted position. $[C]$ is now above $[B]$. $1\,[B] \leftrightarrows 28\,[A]$. The difference of the roots is 3, hence slide the cursor until the difference of opposite scale numbers on $[A]$ and $[B]$ is 3. If this occurs when $7\,[A] \equiv 4\,[B]$ or $4\,[A] \equiv 7\,[B]$, then the roots are 7 and -4.

If the slide projects to the right instead of to the left, $70\,[A] \equiv 40\,[B]$, the roots being 7 and -4.

To solve $x^2 - 11x + 24 = 0$. Set $1\,[B] \leftrightarrows 24\,[A]$. The sum of the roots is 11. Hence slide the cursor until the sum of the numbers on $[A]$ and $[B] = 11$. This occurs when $8\,[A] \equiv 3\,[B]$ or $8\,[B] \equiv 3\,[A]$. The roots are 3 and 8.

11.3 illustrates a slide rule designed to solve any quadratic equation, $x^2 - 2px + q = 0$, the roots being $p \pm \sqrt{(p^2 - q)}$. See "Nomogramme für die komplexen Wurzeln quadratische und reduzierte kubischer Gleichungen", von P. Luckey, *Zeit. für angew. Math. u. Mech.* 13, 1933. [1] and [3] are square scales, to the same scale unit as the uniform scales [2] and [4].

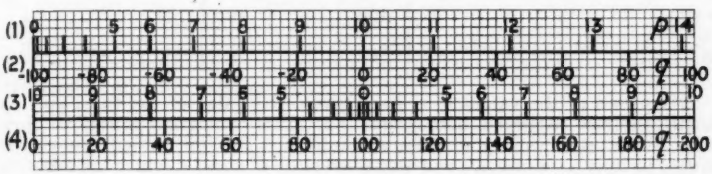
Positive and negative values of p are together on scales [1] and [3], and imaginary numbers are represented to the left of the zero on [3]. $\pm p$ [1] $\equiv p^2$ [4].

To find $\sqrt{(p^2-q)}$, the roots being real.

Set q [2] $\longleftrightarrow$ p [1], then 0 [2] $\equiv \sqrt{(p^2-q)}$ [1].

To find $\sqrt{(q-p^2)}$, the roots being imaginary.

Set p [3] $\longleftrightarrow$ q [4], then 0 [2] $\equiv \pm j\sqrt{(q-p^2)}$ [1].



11.3.

Examples :

$x^2-12x+32=0$; $\sqrt{(p^2-q)}=\pm 2$. The roots are 6 ± 2 or 8 and 4.

$x^2-6x-40=0$; $\sqrt{(p^2-q)}=\pm 7$. The roots are 3 ± 7 or 10 and -4.

$x^2-12x+40=0$; $\sqrt{(p^2-q)}=\pm 2j$. The roots are $6\pm 2j$.

A scale factor of k applied to scales 1 and 3 requires a scale factor of k^2 for 2 and 4.

The same scales will evaluate the following :

$c=a^2+b^2$; 0 [3] $\longleftrightarrow$ a [1] ; then b [3] $\equiv c$ [4].

$c=\sqrt{(a^2+b^2)}$; 0 [3] $\longleftrightarrow$ a [1] ; „ b [3] $\equiv c$ [1].

$c=a^2-b^2$; b [3] $\longleftrightarrow$ a [1] ; „ 0 [2] $\equiv c$ [4].

$c=\sqrt{(a^2-b^2)}$; b [3] $\longleftrightarrow$ a [1] ; „ 0 [2] $\equiv c$ [1].

7. *Failure of Graphical Methods.*

Graphical methods are unsuitable when p/q is either very large or very small. In these cases the method of successive approximation is rapidly carried out using a table of squares such as Barlow's tables.

If p/q is small, the approximate value of the roots is $\pm\sqrt{(-q)}+p$.

Example. $x^2-2x-1000=0$. $(p/q=0{\cdot}001.)$

$p=1$; $q=-1000$; $\pm\sqrt{(-q)}+p=\pm 31{\cdot}6228+1=31{\cdot}62$ or $-30{\cdot}62$.

Let h be the small error in the value of the roots.

$f(x+h)=f(x)+hf'(x)$ or $0=32{\cdot}62^2-65{\cdot}24-1000+65{\cdot}24h-2h$.

$63{\cdot}24h=1065{\cdot}28-1064{\cdot}06=1{\cdot}18$. $h=1{\cdot}18/63{\cdot}24=0{\cdot}0187$.

Thus the roots are 32·639 and $-30{\cdot}639$. If $x=32{\cdot}639$, $x^2-2x-1000=0{\cdot}015$.

If p/q is large, the approximate values of the roots are : $2p$ and $q/2p$.

Example. $x^2-1000x+1=0$ has roots 1000 and 0·001000.

A closer approximation is 999·999 and 0·001000000.

II. Practical Applications of Quadratic Equations.

1. *Optics.*

Consider an illuminated object U and a screen on which a real image V of an object is formed by moving a thin convex lens O between U and V which are kept in a fixed position so that $UV=c=$ constant.

Then $1/u+1/v=1/f$. 1

The magnification $m=v/u$. 2

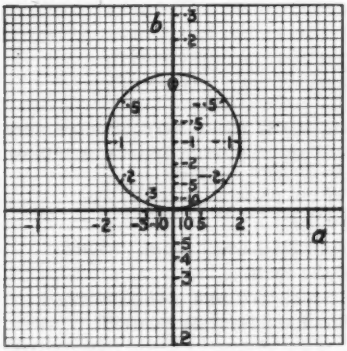
For $f = 20$ cm., the following values of v and m are easily calculated or obtained nomographically.

$f =$ 20 cm.	Virtual images			Real images						
u cm.	10	15	20	24	25	30	40	60	100	120
v cm.	20	60	∞	120	100	60	40	30	25	24
m	2	4	∞	5	4	2	1	0·5	0·25	0·2

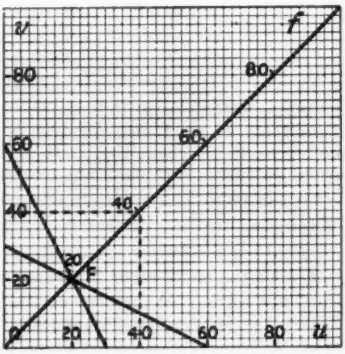
If any two of the values u, v, f are given, the value of the third may be found from the reciprocal nomogram 12.2. This nomogram is easy to draw on squared paper since the scales are uniform. For the present purpose it should be drawn for positive and negative values of u and v, which are negative for virtual objects or images. 12.2 shows that for real objects :

If $u > f$, v is + (indicating a real image).
If $u = f$, $v = \infty$.
If $u < f$, v is − (indicating a virtual image).
If $u = 2f$, $v = 2f$, and $m = 1$.
The minimum value of c for real images is $4f$.

Since $v = c - u$, $1/u + 1/(c-u) = 1/f$,
giving $u^2 - cu + cf = 0$. 3



12.1. $x^3 + ax + b = 0$;
Whittaker's nomogram ; *Calculus of Observations*, p. 129.



12.2. $\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$,
$\frac{1}{c} \cdot u^2 - u + f = 0$, $c = u + v$.

Hence, for each value of c, there are two values u_1 and u_2 of u, with corresponding values v_1 and v_2 of v.

$$v_1 = u_2, \quad v_2 = u_1.$$
$$u_1 + u_2 = v_1 + v_2 = c.$$
$$u_1 \cdot u_2 = v_1 \cdot v_2 = cf, \quad f = u_1 \cdot u_2/c \text{ or } uv/c. \quad \text{.....4}$$
$$u_1 = \tfrac{1}{2}\{c + \sqrt{(c^2 - 4cf)}\}, \quad u_2 = \tfrac{1}{2}\{c - \sqrt{(c^2 - 4cf)}\}.$$

The distance, d, between the two positions of the lens is

$$d = u_1 - u_2 = \sqrt{(c^2 - 4cf)}, \quad f = (c^2 - d^2)/4c. \quad \ldots\ldots\ldots\ldots 5$$

Also, $d = 0$ when $u_1 = u_2$, *i.e.* when the two positions of the lens coincide, and $c = 4f$.

Hence, if $c > 4f$, real images are formed for two positions of the lens.

If $c = 4f$, a real image is formed for coincident lens positions and this is the minimum value of c consistent with the formation of a real image.

If $c < 4f$, no real image is obtained.

Nomogram 12.2 gives the values of u_1 and u_2 (or v_1) for all values of f and c. A secor is rotated about F (at graduation f on the oblique scale) until it cuts the axes of u and v, making $u + v = c$.

2. *Photometry.*

The following example occurs in Whittaker's *Arithmetic of Electrical Engineering*:

"A 16-C.P. lamp is tested against a 10-C.P. standard pentane lamp, the photometer bench being one metre long. Determine the position of the screen when a balance is obtained."

Let x, y be the distances of the lamps from the screen for balance.

$$16/x^2 = 10/(100 - x)^2 \text{ gives } 0{\cdot}6x^2 - 320x + 16000 = 0.$$

This equation reduces to $x^2 - 533{\cdot}3x + 26670 = 0$.

The roots of $x^2 - 5{\cdot}333x + 2{\cdot}6670 = 0$ are 1/100 of the roots of the previous equation, and are equal to 0·5587 and 4·775. Hence the roots of the original equation are 55·87 or 477·5 (this root being inadmissible). Hence $x = 55{\cdot}87$ and $y = 44{\cdot}13$.

An approximate solution of the equation may be found nomographically and more accurate values by successive approximation.

3. *Chemical Equilibrium.*

The equilibrium between alcohol, acetic acid and water, ethyl acetate is expressed by the equation:

$$C_2H_5OH + CH_3COOH \rightleftarrows H_2O + CH_3COOC_2H_5.$$

The velocity constant k_1 for the forward reaction at ordinary temperature is 0·0052 and for the reverse change $k_2 = 0{\cdot}0013$. Hence the equilibrium constant $k = k_1/k_2 = 4$, and varies little with temperature.

If a, b are the initial molecular concentrations (in gram molecules per litre) of alcohol and acetic acid, and x is the molecular concentration of water and ethyl acetate after any time t days, their initial concentrations being zero,

$$v = -dx/dt = k_1(a - x)(b - x) - k_2x^2. \quad \ldots\ldots\ldots\ldots 1$$

v is the velocity of the reaction in the forward direction.

If the initial concentrations of the alcohol and acetic acid are equal,

$$a = b \text{ and } v = k_1(a - x)^2 - k_2x^2. \quad \ldots\ldots\ldots\ldots 2$$

Since $k_1 = 0{\cdot}0052$ and $k_2 = 0{\cdot}0013$,

$$\begin{aligned} v &= 0{\cdot}0052(a^2 - 2ax + x^2) - 0{\cdot}0013x^2 \\ &= 0{\cdot}0039x^2 - 0{\cdot}0104ax + 0{\cdot}0052a^2 \\ &= 0{\cdot}0013(3x^2 - 8ax + 4a^2). \end{aligned}$$

$$dv/dx = 0{\cdot}0013(6x - 8a), \quad dv/dx = 0 \text{ when } x = 4a/3.$$

Since x cannot exceed a, this condition cannot be realised.

Consider the case when the initial molecular concentrations of alcohol and acetic acid are each 1 gram molecule per litre. In this case, $a=b=1$ and $v=0{\cdot}0013(3x^2-8x+4)$.

This gives the following values for x and v:

x	0·0	0·2	0·4	0·6	0·8	1·0
v	0·00520	0·00329	0·00166	0·000364	− 0·000624	− 0·00130

The curve v, x gives $v=0$ when $x=0{\cdot}65$.

If $x'=0{\cdot}65$, and x is the true value,

$$x'=x+h=0{\cdot}65, \quad h=f(x')/f'(x'),$$

$$h=\frac{-0{\cdot}65(8-1{\cdot}95)+4}{3{\cdot}90-8}=-0{\cdot}0165.$$

Hence, $x=0{\cdot}65+0{\cdot}0165=0{\cdot}6665$ or $2/3$.

Thus there is equilibrium when 2/3 of the alcohol and acid have been converted into water and ethyl acetate.

If $a=2$ and $b=3$, or $a=3$ and $b=2$,

$$v=k_1(a-x)(b-x)-k_2x^2=0{\cdot}0052(2-x)(3-x)-0{\cdot}0013x^2$$

$$=0{\cdot}0013\,[4(6-5x+x^2)-x^2]=0{\cdot}0013(24-20x+3x^2).$$

For equilibrium, $v=0$ and $3x^2-20x+24=0$.

This equation may be solved nomographically. The roots are 1·57 and 5·10. The value 5·10 is inadmissible since x could not possibly exceed 2.

If 92 grams of alcohol (molecular weight 46) are heated with 90 grams of acetic acid (molecular weight 60), how much water and ethyl acetate will there be when equilibrium is established? $a=2$, $b=1{\cdot}5$.

For equilibrium, $$k=\frac{x^2}{(2-x)(\frac{3}{4}-x)}=4.$$

Hence, $3x^2-14x+12=0$, giving $x=1{\cdot}13$ and 3·54 (inadmissible).

The molecular weights of water and ethyl acetate being 18 and 88, the equilibrium mixture will contain

$$1{\cdot}13\times 18=20{\cdot}3 \text{ grams of water,}$$

$$1{\cdot}13\times 88=99{\cdot}4 \text{ grams of ethyl acetate.}$$

4. *Dissociation Constant.*

The dissociation constant k for aqueous formic acid is 2×10^{-4} at 15° C. Calculate the degree of dissociation α of a centinormal solution from the formula, $k=\alpha^2/(1-\alpha)V$, $V=100$ litres.

$$\alpha^2/(1-\alpha)100=2\times10^{-4} \text{ or } \alpha^2+(2\times10^{-2})\alpha-(2\times10^{-2})=0,$$

$$\alpha^2+0{\cdot}02\alpha-0{\cdot}02=0, \quad \alpha=0{\cdot}01\pm0{\cdot}142.$$

The negative root is inadmissible. Hence, $\alpha=0{\cdot}152$.

Note. If α is sufficiently small, α may be neglected in comparison with 1 in the term $1-\alpha$ and $k\doteqdot\alpha^2/V$ or $\alpha\doteqdot\sqrt{(kV)}$.

In the present case, $\alpha\doteqdot\sqrt{(2/10^2)}\doteqdot0{\cdot}141$.

For centinormal aqueous hydrocyanic acid $k\doteqdot7\times10^{-10}$, $V=100$ litres. In this case the approximate formula $\alpha\doteqdot\sqrt{(kV)}$ gives almost the same result as the unmodified formula.

5. *Condenser Discharge.*

If a circuit has capacity K, resistance R, and coefficient of self-induction L, and during discharge the potential drop between the plates of the condenser is V at any time t, then

$$L\frac{d^2V}{dt^2}+R\frac{dV}{dt}+\frac{V}{K}=0.$$

The complete solution of such a linear differential equation of the second order may be written down for specified initial conditions from the solution of the corresponding quadratic equation, *e.g.* $L\lambda^2+R\lambda+1/K=0$ or

$$\lambda^2+2(R/2L)\lambda+1/KL=0.$$

The roots of this equation, λ_1, λ_2, are real, equal, or complex, if $R^2K \gtreqless 4L$.

1. $R^2K=4L$ $(\lambda_1=\lambda_2)$. $R=100$ ohm, $K=0{\cdot}2\times10^{-6}$ farad, $L=0{\cdot}0005$ henry.

$$\lambda^2+2(100/0{\cdot}001)\lambda+1/(0{\cdot}2\times10^{-6}\times0{\cdot}0005)=0,$$
$$\lambda^2+2{\cdot}10^5\,.\,\lambda+10^{10}=0.$$

The roots of this equation are 10^5 times those of $\lambda^2+2\lambda+1=0$. Hence,

$$\lambda_1=\lambda_2=-10^5.$$

2. $R^2K>4L$ (λ_1, λ_2, unequal and real). $R=100$ ohm, $K=0{\cdot}2\times10^{-6}$ farad, $L=0{\cdot}0001$ henry,
giving $\lambda^2+2\,.\,5\,.\,10^5\,.\,\lambda+5\,.\,10^{10}=0.$

λ_1, λ_2 are 10^5 times the roots of $\lambda^2+10\lambda+5=0$.

$$\lambda_1=-9{\cdot}472\times10^5,\quad \lambda_2=-0{\cdot}528\times10^5.$$

3. $R^2K<4L$ (λ_1, λ_2, unequal and complex). $R=100$ ohm, $K=0{\cdot}2\times10^{-6}$ farads, $L=0{\cdot}001$ henry.

$$\lambda^2+2\,.\,5\,.\,10^4\,.\,\lambda+50\,.\,10^8=0,$$

λ_1, λ_2 are 10^4 times the roots of $\lambda^2+2\,.\,5\,.\,\lambda+50=0$.

$$\lambda_1=(5+j\,.\,4{\cdot}975)\times10^4,\quad (5-j\,.\,4{\cdot}975)\times10^4.$$

6. *Faulty Barometer.*

A barometer tube of effective length 100 cm. has internal area of cross-section 1 sq. cm. The atmospheric pressure is 76 cm. of mercury. 8 c.c. of air at atmospheric pressure are admitted into the space above the mercury in the tube. Find the pressure of this air if its final temperature is the same as its initial temperature.

Applying Boyle's law, $76\times8=(76-x)(100-x)$ where $x=$ final pressure of the mercury left in the tube.

$$x^2-176x+6992=0.$$

This equation has two positive real roots, α and β, α greater than 100 and inadmissible, and β less than 100.

A graphical method gives the approximate value x' of β as $\doteqdot 60$. If h is the error in this value,

$$h=f(x+h)/f'(x+h)=f(x')/f'(x'),$$
$$f(x')=32,\quad f'(x')=-56,\quad h\doteqdot-0{\cdot}555.$$

Hence, $x=60{\cdot}56$.

7. *Review of Methods.*

A cartesian plot of a quadratic curve exhibits most clearly the general variation of y with x, maximum and minimum values of y and corresponding values of x. The method is laborious and provides only isolated solutions.

Each quadratic function has its own curve. The accuracy of numerical solutions depend on the accuracy with which the curve is drawn.

The geometrical methods of solving quadratic equations commonly presented in elementary courses of mathematics are isolated solutions applicable to small whole number values of p and q. They are usually exercises in illustrating geometrical theorems rather than practical solutions of quadratic equations.

It is remarkable that the elementary graphical methods which have become fashionable are isolated solutions inferior in scope and accuracy to most of the alternative graphical methods of solution. The fashionable technique is to illustrate theorems or to satisfy examination requirements rather than to facilitate computation and provide rapid checks on calculation.

Nomographic methods provide rapid and general numerical solutions. Nomographic methods are available involving the use of straight scales and straight edge only. These methods apply generally to all values of p and q except for p/q large or small. These cases were discussed in Part I, Section 7.

Nomographic methods can be used by beginners in Science and Mathematics and can be invaluable aids to calculation. Abacs could give continuous service for conversion of units, rapid multiplication and division, and for checking the accuracy of observed data in practical work. The average student of twelve years of age can use an addition or subtraction nomogram or abac after two minutes of blackboard demonstration. Similarly, the reciprocal nomogram or abac is used with considerable interest and pleasure by those who have measured u and v for a lens with a view to determining f, and again later for resistances in parallel. Only two minutes' demonstration is necessary.

It is unfortunate that nomographic paper with logarithmic and other scales printed on it ready for use is not available as cheap as squared paper. It could be used even more extensively. It would be a cheap substitute for a slide rule and far more versatile.

C. W. HANSEL.

LONDON BRANCH.

AT a meeting on May 29th, 1943, at the Polytechnic, Regent Street, Mr. F. J. Swan, of Hackney Downs School, opened a discussion on the Future of Examinations.

The speaker sketched the past history and present position of examinations and gave reasons for the continuation of the external test. In all changes it was important to insist that the examination was such as the teachers thought suitable. The proposals of the Cambridge Syndicate were outlined and the suggested syllabus in geometry described.

A long discussion followed, from which it was clear that the meeting felt that there was a danger that fundamental changes might be made before their implications had been fully realised; and further, that the time was ripe for the Mathematical Association to lay down a policy in the matter of the content of mathematical teaching. For example, in geometry, is it desirable

(i) to reduce the bookwork to a small number of theorems?

(ii) to treat geometry and trigonometry together?

(iii) to introduce more practical drawing?

It was decided to devote the next meeting to a discussion of these and similar questions.

A. J. TAYLOR, Hon. Sec.

THE TRAINING OF TEACHERS OF MATHEMATICS TO TECHNICAL STUDENTS.

[The following memorandum, prepared by Dr. Bickley, and adopted after slight modification by the Technical Sub-Committee of the Teaching Committee, has been forwarded to the Board of Education for consideration by their committee on the training of teachers.]

THE recruitment of suitable teachers of ancillary subjects in technical schools and colleges was often difficult before the war, and even if the ideas of the Spens report did not imply a considerable increase in demand, it is time that this supply problem was faced. In the technical subjects themselves the problem is not so acute; the teacher of a branch of engineering is teaching the subject very much as he was taught it, to students whose outlook is very similar to his own. Teachers of physics or chemistry to the same students, although not in quite such a happy position, have certainly in their own studies met with frequent practical applications of their subject, and will usually be entirely sympathetic towards this aspect. With the majority of the teachers of mathematics, however, the position is very different.

It is assumed that full-time professional teachers will, more or less gradually and completely, replace the part-time teachers who have done—and for some time must continue to do—much excellent service. These full-time teachers of mathematics will normally be expected to have obtained an Honours degree in Mathematics, or with Mathematics as one subject in a General Degree, and some post-graduate course of training in teaching. Now although "Applied Mathematics" will almost everywhere be part of the degree course, this term may connote little more than Mechanics, and that of the more abstract kind much closer to "Pure" mathematics than to any technical applications.

Many of our universities have no separate department of Applied Mathematics, and where such departments exist, they are usually much smaller than the corresponding departments of Pure Mathematics. It is seldom that the "pure" mathematician displays wide knowledge of, or indeed much sympathy with, the technical applications of mathematics; often he decries them as sullying the "purity" of his subject. It follows that the majority of those who enter the teaching profession as mathematical specialists have been taught pure mathematics and abstract mechanics in an atmosphere largely ignorant of, if not antagonistic towards, technical applications.

Now while it may be argued that the basic principles and methods of mathematics are the same whether applied, say to the Theory of Numbers, or to the design of a bridge, the technical student generally is not able to apply this basic theory to his own practical problems without guidance and instruction, and can easily be put off—especially in the early stages—by any treatment which does not immediately convince him of the practical importance of the mathematics which he is studying, or by any failure on the part of the teacher to sympathise with his viewpoint. The attitude towards practical application is probably the greatest single factor in the determination of a teacher's success or failure in dealing with technical students.

The successful teacher of mathematics to technical students must therefore fulfil four conditions:

(*a*) he must be a competent *mathematician*;
(*b*) he must be an efficient *teacher*;
(*c*) he must be sympathetic towards the *practical applications* of mathematics;
(*d*) he must possess sufficient *technical knowledge* to be able to present his subject acceptably to his students.

Our present system takes care of (*a*) and (*b*): it seldom encourages (*c*): it gives practically no attention to (*d*).

As regards (*c*) it may not be easy to bring any direct influence to bear upon the departments of Mathematics in the universities, and it is even more unlikely that such influence would radically modify the outlook of the professors in charge. Still, not all such professors are unsympathetic, and some would probably be prepared to cooperate in any proposed scheme for encouraging the supply of teachers of mathematics having the desired outlook. Moreover, indirect influences might arise, as will be mentioned later.

As regards (*d*) there are two aspects, (i) a background of basic technical knowledge into which the incidence of various branches of mathematics upon technology can take its place, and (ii) special knowledge of the needs of students of a particular institution, dictated by local industries.

As regards (i), something could be done in the university course, if students had at that stage already considered entering technical institutions, *e.g.* by partaking in visits to works, etc., with engineering and/or other technical students—and even by arranging occasional visits for themselves. If, during the period of training, the intending teacher of technical students did his "school practice" in a technical college or school, such participation could be even more easily arranged, probably with even greater profit. Then, also, preliminary contacts with the technical departments could be encouraged. But in the undergraduate course, and even in the year at a training college, it is doubtful whether time can be found for more than a superficial survey of technical applications of mathematics. Unless and until full-time terminal or sessional courses in technical applications of mathematics are provided (courses which might attract not only teachers and intending teachers, but also students intending to enter industrial research),* the continuance, and even the extension, of the valuable (and unique) work which was being done at the Oxford Summer School is essential. Summer vacation courses of a week or two's duration can however only touch the fringe of the subject. Valuable as "refresher" courses, and to aid a teacher from a secondary school who intends to change over to a technical one, something much more thorough and sustained is needed for the inexperienced young teacher.

As regards (*d*) (ii), the very important local needs vary enormously from institution to institution, and even from course to course, and from class to class, within any single institution.† Clearly they cannot appreciably be met until a man is appointed to the staff of a particular institution. It is almost essential that, in the first session or so, a newly-appointed teacher's time-table should be lightened to an extent comparable with the amount of such special local technical knowledge which he must acquire in order efficiently to discharge his duties.

Indeed, this principle might be widely extended, and young teachers newly appointed to technical institutions might be required, during their probationary years, and as part of their probation, to make a serious study of one or more important technical subjects;‡ evidence of satisfactory study would

* In this connection, the experiment of Brown University, directed in the first instance towards the researchers, but evoking an unanticipated additional response from the teachers, where the tentative vacation scheme has now blossomed into full-time sessional courses in Applied Mechanics, is notable.

† For a case intermediate between the general and the special local, see the concluding section of a memorandum on *The Post-War Training of Physicists* issued by the Institute of Physics.

‡ *E.g.* in institutions preparing for the London B.Sc. (Eng.) examination, to study say two of the Part I subjects; similarly elsewhere.

be demanded, while the teaching time-table would, of course, be proportionately lightened.

Finally, there is the underlying problem of attracting mathematical students and teachers to this essential work of training our technicians. For the right type, the work has an interest far transcending that, say, in a secondary school, but this aspect is rarely encountered by the undergraduate mathematician. He only sees that in most technical schools and colleges evening work is demanded, and realises that some additional effort will be called for to amass the technical knowledge necessary to the satisfactory performance of the work. To offset the inconvenience and additional strain of evening work (present even when the total number of teaching hours is no more than in a day school), or to reward the additional study, he sees no financial inducement, or enhanced prospects.

On the contrary, it is time that clear recognition be given to the fact that the teacher of mathematics to technical students must, if he is to be efficient and successful, possess qualities and knowledge additional to and different from those which would fit him for work in, say, a day secondary school. The status, emoluments, conditions of work, and prospects of the teachers of mathematics in technical institutions should be so improved that students of mathematics may at an early stage envisage, and commence to prepare themselves for, a career in such institutions. The provision, for such students, of special facilities such as have been outlined, is also necessary, but the existence of a body of these students in any numbers would be the strongest incentive to and guarantee of the speedy and complete provision of these special facilities.

W. G. Bickley
February, 1943. (Chairman, Technical Sub-Committee).

SHEFFIELD AND DISTRICT BRANCH.

Report for the Session, 1942–1943.

The Branch has again held three well attended meetings, one in each term.

That on June 30th, 1942, the fourth annual general meeting, took place at High Storrs Grammar School for Girls by the kind invitation of Miss Battensby (Headmistress) and her Staff. The Officers elected were: President, Miss Holdsworth; Vice-Presidents, Dr. Stewart, Mr. Smith; Committee, Miss May, Miss Woodland, Mr. Cox, Mr. Cunnington, Dr. Thompson, Mr. G. W. Wilkinson; Hon. Sec. and Treas., Mr. Cowley. Following the business, three short papers of lively interest were given. Professor Daniell and Mr. Smith dealt with the inter-relations of University and Sixth Form Mathematics and Dr. Thompson reviewed the mathematical basis of Intelligence Testing.

By the courtesy of the University, the meeting on Sept. 22nd, 1942, was held in the Mathematics Lecture Room. Dr. Oldham (Reading University) lectured on "A psychological study of mathematical ability with special reference to school mathematics"; the lecture embodied the results of her own researches over a period of several years. On March 18th, 1943, in the University, Professor Brodetsky (Leeds University) gave an illustrated lecture on "Aeroplane Mathematics".

The present membership of the Branch is about 55. Expenditure for the year 1942 exceeded income by £3 4s. 9d., but the Hon. Treasurer is glad to say that arrears of subscription are being paid and that the Branch is still solvent.

The Branch is deeply indebted to its lecturers and to everyone who has contributed to its smooth pleasant running. John W. Cowley (Hon. Sec.).

CORRESPONDENCE.

SCHOOL CERTIFICATE MATHEMATICS.

To the Editor of the *Mathematical Gazette*.

SIR,—As the Teaching Committee has not yet had the opportunity of meeting to deliberate on the matter raised by Mr. Brereton's letter (*Gazette*, p. 80), will you allow me to reply to it as an ordinary member of the Association?

I wish in the first place to welcome the move taken by the Cambridge Local Syndicate. I feel that the changes are entirely on the right lines, although it may be possible to criticise some of the details. I feel that it may be desirable for the Syndicate to go a step farther by revising the requirements of the other two papers. It is a doubtful point whether there ought to be three separate papers, on Arithmetic, on Algebra, and on Geometry and Trigonometry, in a school certificate examination. It has been generally recommended by Inspectors in the schools that as many boys as possible ought to cover some of the ground of what is usually called "Additional Mathematics". It might be an advantage if some of the "additional" work was tested in the "elementary" papers, possibly by means of alternative questions giving a wide choice; and to make this effective, some of the less useful parts of the "elementary" syllabus might be purged.

It seems therefore that there is ample work for some Committee of the kind suggested by Mr. Brereton. It may be difficult to form a strong representative committee at the present time when so many of the younger mathematicians are on service. But we ought to make the effort to place ourselves in a position to take up this task as soon as the opportunity offers. It is hoped that a meeting of the Teaching Committee will be held in August. It will be useful if members of the Association will make their views known by writing to the Secretary or to the *Gazette*.

Yours faithfully, A. ROBSON.

To the Editor of the *Mathematical Gazette*.

SIR,—The letter from Mr. Brereton, and the new Cambridge syllabus and specimen paper are so important that they deserve the fullest consideration by members of our Association.

It is unfortunate that the Association was not asked to help in preparing this new syllabus until a late stage. Since the outbreak of war most of the Committee work of the Association has been suspended, but in 1939 a newly-formed Examinations Sub-Committee had begun revision of School Certificate syllabuses. It had not considered geometry because it became clear that no subject could be considered apart from others, or from work at a higher stage. Discussion by the whole Association of some of the problems which had revealed themselves was essential. It is interesting to mention these problems because they are not solved by the new Cambridge syllabus.

The first problem arises if we teach arithmetic and the other subjects as related parts of elementary mathematics and yet examine them by separate papers. Trigonometry now enters into arithmetic and geometry, and so the Cambridge syllabus attempts to meet this by adding further questions on trigonometry in the *geometry* paper. Is it not time to move forward and to set two or three "mixed" papers in "Elementary Mathematics"? The arrangement of the papers set by the Oxford and Cambridge Joint Board may be familiar to members, and, with modifications, might be preferable to "separate subject" papers. Even if the new Cambridge arrangement of

questions is approved, would it not be better to restrict the trigonometry questions in the arithmetic paper to computation exercises? And in the geometry paper to set questions depending on a diagram or on an idea of some interest? Small numbers or literal symbols could be used, and exercises might be set on navigation, the earth, simple map projections, solid geometry. This kind of trigonometry appears in problems in mechanics and deserves consideration.

The second problem arises through the blurring of the lines that used to separate the three stages, "school certificate", "more advanced mathematics" and "higher certificate", through changes in recent years. Perhaps it is more necessary to reconsider algebra (including calculus) at these stages, than geometry. But it is very desirable to consider the higher certificate geometry syllabus before restating the school certificate syllabus.

Since 1939 members of the Examinations Sub-Committee have been too busy and too widely separated to take up the considerable task awaiting them, but the formation of such a fully representative body as Mr. Brereton suggests in his letter would be a most valuable forward step.

Some comments may be made on details of the new Cambridge syllabus:

(i) The inclusion of A.T.C. work and of "plan and elevation" is most welcome, but the "drawing-board" section of the syllabus should be expanded in detail. The illustrative question is too simple to give much guidance.

(ii) The treatment of theorems will give rise to argument. It is good to aim at cutting down memory work but it is a pity to make the sequence rigid. Is it necessary in a geometry syllabus to state a detailed list of theorems? If there *is* a list, long or short, it may encourage memory work. The shorter the list, the greater the concentration and the certainty that one or more of the theorems learnt will appear. Why not state properties or theorems for which formal proofs are *not* required, and see that examiners test the inevitable major theorems by framing questions with reference to a printed diagram, or a given particular enunciation as well as in the conventional formal way?

(iii) The interpretation of a syllabus by examiners is more important than the syllabus itself. It is refreshing to find the Cambridge Local Syndicate taking so sensible an attitude in the matter of textbooks: by suitable questions on the new syllabus a most useful stimulus can be given to sound teaching. But the form of the examination paper needs reconsideration. It seems, psychologically, formidable and the candidates' task of selection of questions rather hard.

Yours, etc., C. T. DALTRY.

A HAMILTON STAMP.

To the Editor of the *Mathematical Gazette*.

SIR,—Philatelists in the Association may be interested to know that it is said that the Irish Government is proposing to issue postage stamps bearing a portrait of Sir William Rowan Hamilton. Has any other great mathematician been honoured in this fashion?

Yours, etc., J. RIVERSDALE COLTHURST.

THE PROBLEM BUREAU.

MEMBERS are reminded that the Problem Bureau is still in existence under the direction of Mr. A. S. Gosset Tanner, 115 Radbourne Street, Derby.

MATHEMATICAL NOTES.

1664. *Notes on Conics.* 8 : *Phantom powers and Frégier points.*

1. What traces remain of the powers of a point O' for a conic as O' tends to a point O on the conic? The powers all vanish, but since the ratio of the powers in two fixed directions is independent of the position of O' in the plane, they vanish in definite proportions, and these proportions should be recognisable from the limiting point O itself.

And so they are. Let OP be a chord of the conic, and let a line through O' parallel to OP cut the conic in P' and Q', where, as O' tends to O, it is P' that tends to P and Q' that tends to O. Then if $O'P'$ cuts the tangent at O in T', the ratio $O'Q'/O'T'$ tends to unity. Hence for any two chords OP_1, OP_2, the ratio of the powers $O'Q'_1 \cdot O'P'_1$, $O'Q'_2 \cdot O'P'_2$ tends to the same limit as the ratio of the products $O'T'_1 \cdot O'P'_1$, $O'T'_2 \cdot O'P'_2$. But if the lines $O'P'_1$, $O'P'_2$ cut a fixed line parallel to the tangent at O in U'_1, U'_2, we have always $O'T'_1/O'U'_1 = O'T'_2/O'U'_2$, and therefore the ratio of the products $O'T'_1 \cdot O'P'_1$, $O'T'_2 \cdot O'P'_2$ is the same as the ratio of the non-vanishing products $O'U'_1 \cdot O'P'_1$, $O'U'_2 \cdot O'P'_2$.

Th. 1. *If a variable line through a fixed point O of a conic cuts the conic again in P and cuts a fixed line parallel to the tangent at O in U, the product $OU \cdot OP$ is proportional to the power of an arbitrary fixed point not on the conic along the line through that point parallel to OP.*

Direct proofs of this important theorem are readily constructed.

(i) If the conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0,$$

the power of a fixed point in the variable direction (l, m) is inversely proportional to $al^2 + 2hlm + bm^2$. On the other hand, the radius from the origin to the conic in this direction is

$$-2(gl + fm)/(al^2 + 2hlm + bm^2),$$

and the radius from the origin to the line $gx + fy + k = 0$ in the same direction is $-k/(gl + fm)$.

(ii) Let OP cut the directrix in R ; the line through O parallel to PS cuts SR in p, the point on the eccentric circle of O which corresponds to P on the

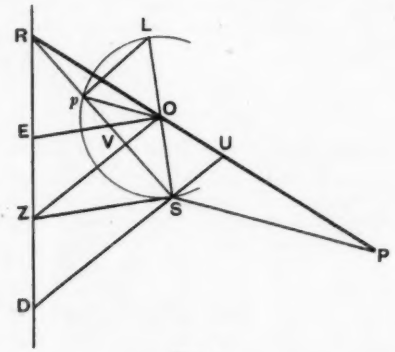


FIG. 1.

conic, and we have as usual $OP/RO = Sp/pR$. The tangent at O to the conic cuts the tangent at S to the circle in a point Z on the directrix, and if the line through S parallel to OZ cuts the directrix in D and cuts OP in U, then $OU/RO = DZ/ZR$. The product we have to examine is therefore

$$Sp \,.\, DZ/pR \,.\, ZR,$$

and we must expect to relate this product to $pR \,.\, SR$, the power of R for the eccentric circle, since it is this power which in general enters into the expression for the power of a point for the conic. Let the line through O parallel to SZ cut the directrix in E and cut SR in V. Then, on the one hand, $ZR/ZE = SR/SV$. On the other hand, since SOV is a right angle, $SV \,.\, Sp = SO \,.\, SL$, where L is diametrically opposite to S on the eccentric circle; that is, $SV \,.\, Sp = 2SO^2$.

Combining the equalities, we have

$$OU \,.\, OP = \frac{2(DZ/ZE)\,SO^2}{Rp \,.\, RS/RO^2}.$$

This is the formula to be compared with the familiar

$$OP \,.\, OQ = \frac{Sp \,.\, Sq}{Rp \,.\, Rq/RO^2}.$$

The denominator is the same, for when O is on the conic, S plays the part of q for all chords OPQ through O, and the numerator is in each case independent of the position of R on the directrix.

(iii) A much shorter proof is due to Bennett (Milne and Davis, *Geometrical Conics*, p. 195, but without recognition of the theorem as a limiting form). Taking the fixed line as the diameter parallel to the tangent at O, let the

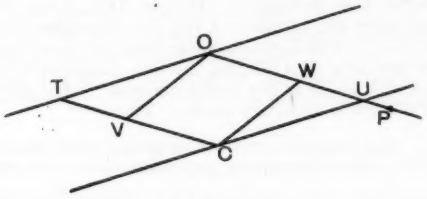


FIG. 2.

diameter parallel to OP cut the tangent at O in T, let the diameter conjugate to CT cut OP in W, and let the line through O parallel to CW cut CT in V. Since CW bisects all chords parallel to CT, we have

$$OU \,.\, OP = 2OW \,.\, OU = 2CV \,.\, CT,$$

and this is the result to be established, for OV is the ordinate from O to the diameter CT, and therefore $CV \,.\, CT$ is the negative of the power of the centre C along this diameter, whether or not the diameter cuts the curve.

2. At any fixed point not on the conic, the sum of the reciprocals of powers in perpendicular directions is constant. It follows that, if O is a fixed point on the conic, and if two perpendicular chords OP, OQ cut a fixed line parallel to the tangent at O in U, V, then $1/OU \,.\, OP + 1/OV \,.\, OQ$ is constant. Let the normal at O cut UV in K, let KX, KY be the perpendiculars from K to

OP, OQ, and let the line through K parallel to PQ cut OP, OQ in P', Q'. Then since UKV is perpendicular to OK, we have $OK^2 = OX \cdot OU = OY \cdot OV$, and therefore

$$\frac{1}{OU \cdot OP'} + \frac{1}{OV \cdot OQ'} = \frac{1}{OK^2}\left(\frac{OX}{OP'} + \frac{OY}{OQ'}\right) = \frac{1}{OK^2}$$

and is constant. But if OK cuts PQ in F, then

$$\frac{1}{OU \cdot OP'} + \frac{1}{OV \cdot OQ'} = \frac{OF}{OK}\left(\frac{1}{OU \cdot OP} + \frac{1}{OV \cdot OQ}\right).$$

Hence OF/OK is the ratio of one constant to another, and F, like K, is a fixed point. This is Frégier's theorem :

Th. 2. *The chords of a conic which subtend a right angle at a fixed point of the conic pass through a fixed point on the normal at that point.*

This theorem is an immediate deduction from the general theory of ranges in involution on a conic, and it can be established in a few words by reciprocation or by the use of coordinates. The object of this paragraph is not to challenge the simplicity of these well-known proofs, but to exhibit what is after all a metrical theorem in as direct a relation as possible to a fundamental metrical property of the conic.

3. The Frégier point enters almost dramatically into the geometry of the pedal circle of a point for a triangle. Let the normal to a conic at O cut the conic again in E, and let OP, OQ be two perpendicular chords ; then PQ cuts OE in F, the Frégier point of O, and the intersections (OP, EQ) and (OQ, EP) are conjugate points on the polar of F. That is, the polar of F is a line f such that two points on f are conjugate if the lines joining them to O are perpendicular ; in other words, if f cuts the conic in U, V, then OU, OV are the isotropic lines through O. If now ABC is any triangle inscribed in the conic, then since the vertices of the two triangles ABC, OUV are on a conic, there is a conic which touches the sides of these two triangles, that is, which touches the sides of ABC and the line f and has O for a focus. The auxiliary circle of this conic is the pedal circle of O for ABC, and the projection of O on f is on this auxiliary circle. Thus

Th. 3. *If O is any point of a conic, the pedal circle of O for any triangle inscribed in the conic passes through the projection of O on the polar of the Frégier point of O.*

To establish this theorem, which is an important one, directly in real geometry, let L be the point of intersection of EP and OQ, a point which, we have seen to be on f, the polar of F, let G be the projection of O on f, or, as we may say briefly, the Frégier projection of O, and denote by Γ the circle round O through G. Since the three points E, P, L are collinear, their polars e_Γ, p_Γ, l_Γ for Γ are concurrent. But since L is on f, which is the tangent to Γ at G, the polar l_Γ of L for Γ passes through G, and since OL is perpendicular to OP, l_Γ is parallel to OP and is perpendicular to p_Γ. In other words, the intersection of the polars l_Γ, p_Γ is the projection of G on p_Γ, and this projection is on the line e_Γ ,whatever the position of P on the conic.

If, then, A, B, C are any three points of the conic, the projections of G on the polars a_Γ, b_Γ, c_Γ are collinear points, and therefore G is concyclic with the vertices of the triangle whose sides are these three polars. But the intersection of the polars b_Γ, c_Γ is the pole of BC and is the inverse for Γ of the projection of O on BC. Since the inverses of four concyclic points are them-

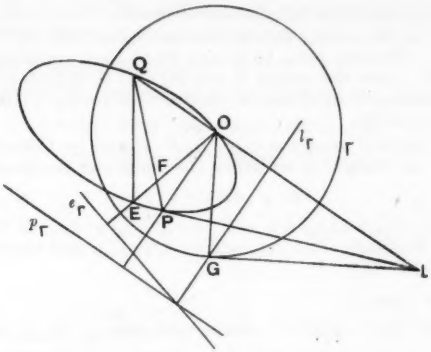


FIG. 3.

selves concyclic, and G is self-inverse for Γ, it follows that G is concyclic with the projections of O on BC, CA, AB.

We can phrase the argument of this proof somewhat differently. Since O, the centre of Γ, is a point of the original conic, the reciprocal of this conic for Γ is a parabola, and the first stage of the proof identifies G with the focus of this parabola. The concluding stage inverts the circumcircle of a tangent triangle. But this version of the proof is only an unnecessary elaboration, for the properties of the parabola which are involved express ultimately the very pedal-line theorem which we have used explicitly.

4. If we think not of a given conic but of a given triangle, the form of Th. 3 is changed. With ABC and O given, the pedal circle of O for ABC is a definite circle, and there is a pencil of conics through the four points ; to each of these conics Th. 3 is applicable : the Frégier projection of O for each conic of the pencil $OABC$ is on the pedal circle of O for ABC. Moreover, it is a simple matter to reverse the steps of the construction, determining E, and therefore the conic $OABCE$, from G, provided only that G is on the pedal circle. Hence

Th. 4. *The pedal circle of a point O for a triangle ABC is the locus of the Frégier projections of O for conics of the pencil $OABC$.*

This description of the pedal circle as a locus is the straightforward equivalent of the description given by Baker (*Introduction*, p. 116) by means of pencils whose vertices are the circular points. E. H. N.

1665. *On Note* **1605.**

Surely a much more natural reason for the conventions of sign employed in the trigonometry of angles of all magnitudes is to be found in the formulae of transformation from polar to cartesian coordinates? With the accepted convention of rectangular cartesian axes, the equations $x = r\cos\theta$, $y = r\sin\theta$ will be true in all quadrants if and only if r is considered always positive ; it is this universal result which enables formulae such as the addition formulae of trigonometry to be proved more simply for angles of all magnitudes, by the use of analytical geometry.

Any other convention would, by the way, give discontinuities at finite points in some at least of the trigonometrical functions. F. M. GOLDNER.

1666. *Two analytical proofs of Pascal's Theorem.*

(i) Let $s=0$ be the conic, 123456 the hexagon, $(12)=0$ the join of the vertices 1, 2, etc. The line-pairs 12,56 and 23,45 have equations $s=k(25)(16)$ and $s=l(25)(34)$. Also the points X (12,45) and Y (23,56) lie on them, and they do not lie on $s=0$ or $(25)=0$; hence they lie on $k(16)=l(34)$ which passes through Z(16,34).

(ii) Let X, Y (above) meet the conic in A, B and let C be the pole of AB. Then with ABC as triangle of reference the conic can be taken as

$$x : y : z = t^2 : 1 : t.$$

The chords $x-(t_1+t_2)z+t_1t_2y=0$ and $x-(t_4+t_5)z+t_4t_5=0$ meet on $z=0$; hence $t_1t_2=t_4t_5$. Similarly $t_2t_3=t_5t_6$ and so $t_1t_3=t_4t_6$, and therefore the chords 16,34 meet on XY. A. ROBSON.

1667. *On Note* 1569.

If the chord of $x^2/a^2+y^2/b^2=1$ whose mid-point is (x_1, y_1) is to be derived from

$$(x-h)(x-h')/a^2+(y-k)(y-k')/b^2=x^2/a^2+y^2/b^2, \quad \ldots\ldots\ldots\ldots\text{(i)}$$

this should be done by writing the equation in the form

$$x(h+h')/a^2+y(k+k')/b^2=\text{constant}$$

or

$$xx_1/a^2+yy_1/b^2=\text{constant}=x_1^2/a^2+y_1/b^2.$$

The result of dividing by y_1/b^2 as on p. 51 of the *Gazette*, thus obtaining the slope, is that it becomes necessary in lines 2,3 of p. 52 to multiply by y_1/b^2.

The excuse for using equation (i) is that it avoids any assumption about the polar. In dealing with the general equation of the second degree, as in the later part of Note 1569, it can hardly be necessary to avoid the polar unless some examination has very peculiar regulations and some teacher is afraid to go outside the syllabus.

The method of equation (i) was given in Salmon's *Conic Sections* nearly a hundred years ago. When Salmon's masterpiece was first published it was perhaps fifty years ahead of its time, but it was scarcely a hundred years ahead. Certain improvements have been introduced into elementary analytical geometry since 1850. For example, for about the last fifty years it has been regularly explained to undergraduates at Cambridge, and in particular in lectures at Emmanuel College, that it is convenient to denote

$$ax_1x_2+h(x_1y_2+x_2y_1)+by_1y_2+g(x_1+x_2)+f(y_1+y_2)+c$$

by s_{12}, and to denote the same expression with the suffix 2 omitted by s_1, and the same expression with the suffix 2 changed into a 1 by s_{11}, and so on. This notation has been explained to boys of 15 in at least one school for the past thirty years. It has appeared in an Oxford scholarship paper. There is no difficulty about it, and its merits are obvious. Why is it that contributors to the *Gazette* will not use it?

The notation is something more than a mere convenient shorthand. It amounts to a kind of calculus. In using it, the symbols do not have to be constantly replaced by their expanded values. The best applications are those to the polar theory arising from Joachimsthal's equation

$$s_{11}k_1^2+2s_{12}k_1k_2+s_{22}k_2^2=0,$$

and the chord with mid-point (x_1, y_1) being parallel to $s_1=0$ is $s_1=s_{11}$. But, even without the polar, $s_1+s_2=s_{12}$ is the chord $(x_1, y_1)(x_2, y_2)$, and if

$$x_1+x_2=2x_3 \text{ and } y_1+y_2=2y_3$$

it follows that $s_1 + s_2 = 2s_3$, and so the chord is $s_3 = \text{constant} = s_{33}$. Thus the use of the artificial equation (i) can be avoided. A. R.

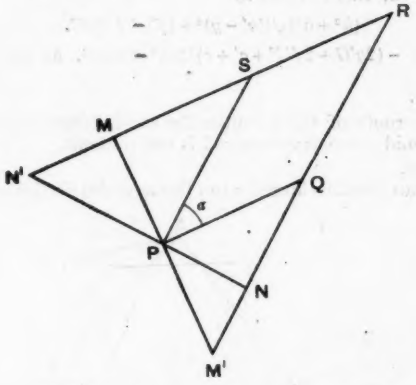
1668. *The area of a parallelogram.* —

To find the area of the parallelogram whose sides are given by

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and

$$S' \equiv ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0.$$



1. In the figure, $PQRS$ is the parallelogram, MPM' is perpendicular to PQ and NPN' is perpendicular to PS.

Let $P(x_1, y_1)$ be the point of intersection of the lines $S = 0$ and let the angle QPS be α. The combined equation of MPM' and NPN' is

$$T \equiv b(x - x_1)^2 - 2h(x - x_1)(y - y_1) + b(y - y_1)^2 = 0$$

and the combined equation of RQ and RS is

$$S' \equiv ax^2 + 2hy + by^2 + 2g'x + 2f'y + c' = 0.$$

Hence the equation of the circle $MNM'N'$ is

$$S' + T \equiv (a + b)(x^2 + y^2) + \ldots = 0.$$

Hence $MP \cdot PM' = S_1'/(a + b)$.

The area required $= PQ \cdot PS \cdot \sin\alpha = PM \cdot PN \operatorname{cosec}\alpha$

$= PM \cdot PM' \cdot \cot\alpha = S_1'/2\sqrt{(h^2 - ab)}$.

Now $S_1' = x_1(ax_1 + hy_1) + y_1(hx_1 + by_1) + 2g'x_1 + 2f'y_1 + c'$

$= 2g'x_1 + 2f'y_1 + c' + c$, since $\Delta = 0$,

$$= 2 \begin{vmatrix} a & h & g \\ h & b & f \\ g' & f' & \frac{1}{2}(c + c') \end{vmatrix} \div (ab - h^2).$$

Hence the area is

$$- \begin{vmatrix} a & h & g \\ h & b & f \\ g' & f' & \frac{1}{2}(c + c') \end{vmatrix} \div (h^2 - ab)^{3/2}.$$

2. For a rectangle $(a+b=0)$ this method fails, but we can use the fact that the diagonals are equal. The equation of QS is

$$2(g'-g)x+2(f'-f)y+(c'-c)=0$$

and the length of the perpendicular from P on it is

$$\{2(g'-g)G+2(f'-f)F+(c'-c)C\}/2C\{(g'-g)^2+(f'-f)^2\}.$$

Also in this case $QS=PR=\sqrt{\{(G'-G)^2+(F'-F)^2\}}/C$,

and putting $b=-a$, this reduces to

$$(h^2+a^2)\sqrt{\{(g'-g)^2+(f'-f)^2\}}/C.$$

Hence the area is $-(2g'G+2f'F+c'+c)/2(h^2-ab)^{3/2}$, as before.

N. M. GIBBINS.

1669. *Sundials.*

The following proofs of the formulae for sundials avoid spherical trigonometry, and the solid geometry involved is not difficult.

Horizontal sundial.

To prove that tan (shadow angle) = tan (hour angle) sin (latitude).

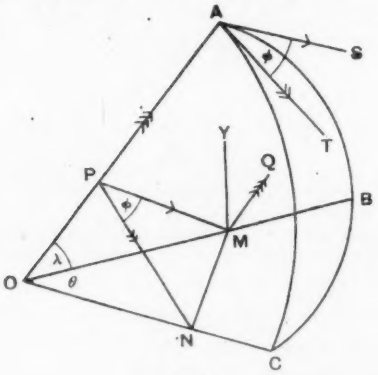


FIG. 1.

The style OA points to the celestial north pole, OB is true north, OC is the edge of the shadow of OA, λ the latitude of O, ϕ is the hour angle of the sun and θ the shadow angle. AB, BC, CA are arcs of great circles on a sphere centre O, AS and AT are tangents to AB and AC at A. PM is drawn parallel to AS, PN to AT, MQ to PA, and MY is vertical.

By construction PA is perpendicular to the plane PMN; hence MQ is perpendicular to the plane PMN and hence the angle $NMQ=90°$. Also YM is perpendicular to the horizontal plane OMN, and hence the angle $NMY=90°$. NM is therefore perpendicular to two lines in the plane QMY. Thus NM is perpendicular to the plane QMY, and so in particular the angles OMN, PMN are each 90°. Further, by parallels the angle $MPN=\phi$. Thence by inspection of the figure

$$\begin{aligned}\tan\theta &= MN/OM\\ &= (MN/PM)(PM/OM)\\ &= \tan\phi\sin\lambda.\end{aligned}$$

Sundial on a vertical wall facing South.

To prove that tan (shadow angle) = tan (hour angle) cos (latitude).

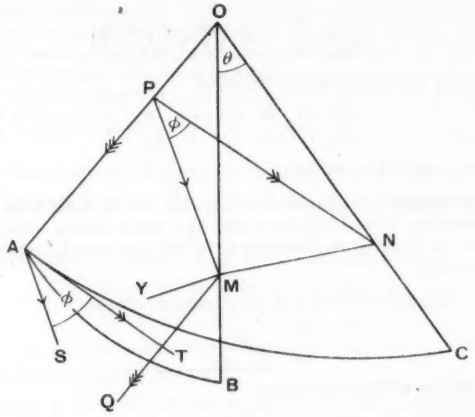


FIG. 2.

The style OA points to the celestial south pole, OB is the downward vertical and OC the edge of the shadow of the style. The angle SAT is the hour angle, the angle AOB is the complement of the latitude. PM is drawn parallel to AS, PN to AT, MQ to PA, and MY is horizontal in the plane AOB.

As before, the angle $NMQ = 90°$, the angle $NMY = 90°$. Therefore NM is perpendicular to the plane QMY, and this plane contains $AOPB$ so that the angles OMN, PMN are each 90°. By inspection

$$\begin{aligned}\tan\theta &= MN/OM\\ &= (MN/PM)(PM/OM)\\ &= \tan\phi \sin(90° - \lambda)\\ &= \tan\phi \cos\lambda.\end{aligned}$$

JOHN W. HESSELGREAVES.

1670. *A note on envelopes.*

The theory of envelopes, as developed in the standard textbooks on the Calculus, is concerned with a family of curves given by an equation of the form $f(x, y, \alpha) = 0$, where α is a variable parameter. I am unable to find any reference to the case where a typical member of the family is given in the parametric form $x = x(\alpha, t)$, $y = y(\alpha, t)$, the parameter α being fixed for each particular curve and t varying as we describe this curve. Some results in this direction may therefore be of interest.

In order to find the envelope of a family of curves of the form under consideration we may take α as a parameter which varies as we describe the envelope. To determine parametric equations for the envelope we then have to determine the value of t, as a function of α, for which a typical curve of the family touches the envelope. Substituting this value of t in the original equations of the family will then give us the required equations of the envelope.

The equation determining t as a function of α is obtained as follows. At the point of contact of a typical curve of the family with the envelope, the slope of the curve is $\dfrac{\partial y}{\partial t}\Big/\dfrac{\partial x}{\partial t}$, while the slope of the envelope is

$$\left(\frac{\partial y}{\partial \alpha}+\frac{\partial y}{\partial t}\cdot\frac{dt}{d\alpha}\right)\Big/\left(\frac{\partial x}{\partial \alpha}+\frac{\partial x}{\partial t}\cdot\frac{dt}{d\alpha}\right).$$

Equating these two expressions we have

$$\frac{\partial(x, y)}{\partial(\alpha, t)}\equiv\frac{\partial x}{\partial \alpha}\cdot\frac{\partial y}{\partial t}-\frac{\partial x}{\partial t}\cdot\frac{\partial y}{\partial \alpha}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and this is the required equation.

Many of the standard examples of envelopes can be dealt with by the above method. Examples 1 and 2 below are two such cases. Example 3 is of special interest, as it gives an instance in which the envelope itself consists of a family of curves.

Example 1. *Find the envelope of the family of ellipses*

$$\frac{x^2}{\alpha^2}+\frac{y^2}{(c-\alpha)^2}=1, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where c is a positive absolute constant.

The ellipse (2) may be written, in parametric form,

$$x=\alpha\cos t, \quad y=(c-\alpha)\sin t.$$

Equation (1) then gives $\quad (c-\alpha)\cos^2 t=\alpha\sin^2 t,$

or $$\tan^2 t=(c-\alpha)/\alpha.$$

Hence the required envelope is

$$x^2=\alpha^3/c, \quad y^2=(c-\alpha)^3/c,$$

or $$x^{2/3}+y^{2/3}=c^{2/3}.$$

Example 2. *Find the envelope of the family of semicubical parabolas*

$$(y+\alpha)^2=\alpha x^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Writing $\alpha=\beta^3$ and taking β for a new parameter, the curve (3) is given by the equations

$$x=t^2/\beta, \quad y=t^3-\beta^3.$$

The equation $\dfrac{\partial(x, y)}{\partial(\beta, t)}=0$ now gives either $t=0$ (the line of cusps) or $t^3=2\beta^3$. We therefore have, for the true envelope,

$$x^3=4\beta^3, \quad y=\beta^3,$$

or $$4y=x^3.$$

Example 3. *Find the envelope of the family of spirals*

$$x=\alpha\cos \alpha t, \quad y=t\sin \alpha t.$$

Equation (1) gives, on reduction,

$$\tan 2\alpha t=-2\alpha t.$$

Now the equation $\quad \tan 2u=-2u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

has an infinity of solutions and it is easily verified that for each such solution (except $u=0$) the rectangular hyperbola

$$xy=\tfrac{1}{4}u\sin 2u \qquad \ldots\ldots(5)$$

is a true envelope of the family. The complete envelope thus consists of the family of all such rectangular hyperbolae. It is also easily verified that if $u\to\pm\infty$, always satisfying (4), then the right-hand side of (5) is alternately positive and negative and tends, in absolute value, to infinity.

The solution $u=0$ to (4) gives $t=0$ and so $y=0$, which is not a true envelope, but a line of " stop points ".

D. R. DICKINSON.

1671. *The triangle whose sides are the roots of* $x^3-px^2+qx-r=0$.

The triangle whose sides are the roots of the cubic equation

$$x^3-px^2+qx-r=0$$

presents some points of interest. If a, b, c be the sides, then

$$a+b+c=p,\ bc+ca+ab=q,\ abc=r.$$

Also
$$a^2+b^2+c^2=p^2-2q=2P, \text{ say,}$$

and
$$b^2c^2+c^2a^2+a^2b^2=q^2-2pr=Q.$$

Elementary considerations show that :

(i) The triangle is isosceles if two roots of the given equation are equal, that is, if
$$27r^2-18pqr+4q^3+4p^3r-p^2q^2=0.$$

(ii) The triangle is equilateral if
$$p^2=27r \text{ and } p^2=3q.$$

(iii) The triangle is right-angled if
$$(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)=0,$$

that is, if
$$(P-a^2)(P-b^2)(P-c^2)=0,$$

or
$$P^3-2P^2+P(q^2-2pr)-r^2=0.$$

This gives
$$r^2=P(q^2-2pr-P^2)$$
$$=P(-\tfrac{1}{4}p^4+p^2q-2pr)$$

or
$$8r^2=(p^2-2q)(-p^4+4p^2q-8pr).$$

If Δ be the area of the triangle, then

$$16\Delta^2=(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$=p(p-2a)(p-2b)(p-2c)$$
$$=p(-p^3+4pq-8r).$$

Hence (iv) the triangle is real, of zero area, or imaginary according as

$$p^3+8r<,\ =\ \text{or}\ >4pq.$$

As a further example we may show that (v) the squares of the medians are the roots of the equation

$$64x^3-96Px^2+36Qx+32P^3-36PQ+27r^2=0.$$

For, if m_1, m_2, m_3 be the medians, then

$$m_1^2=-\tfrac{1}{4}a^2+\tfrac{1}{2}b^2+\tfrac{1}{2}c^2=P-\tfrac{3}{4}a^2,$$

and
$$m_2^2=P-\tfrac{3}{4}b^2,\ m_3^2=P-\tfrac{3}{4}c^2.$$

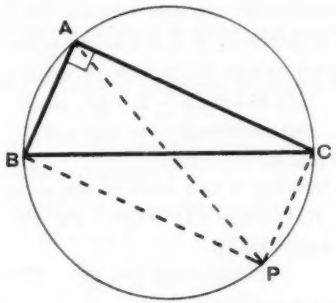
Hence
$$m_1^2+m_2^2+m_3^2=3P/2,$$
$$m_2^2m_3^2+m_3^2m_1^2+m_1^2m_2^2=9Q/16,$$
$$m_1^2m_2^2m_3^2=-(32P^3-36PQ+27r^2)/64,$$
and so m_1^2, m_2^2, m_3^2 are the roots of the above equation. W. H. SALMON.

1672. *A geometrical problem.*

The lengths of the edges of three cubical blocks are respectively 3″, 4″ and 5″: how can the blocks be divided into parts which fit together to form a cube having edges 6″ in length? $(3^3+4^3+5^3=6^3.)$ H. W. R.

1673. *On Notes* 1550, 1597.

"If Pythagoras is to come after similar triangles", why not drag in Ptolemy's theorem also?



Complete the rectangle $ABPC$ and join AP. The four points A, B, C, P must be concyclic and hence
$$AC\,.\,BP+AB\,.\,CP=BC\,.\,AP.$$
But $AC=BP$, etc. Thus $AC^2+AB^2=BC^2$.

I do not advocate the teaching of this proof, though I have employed the method of Note 1597 with success for some years. R. W. MORRIS.

PLYMOUTH AND DISTRICT BRANCH.

IN the past twelve months three extremely successful meetings have been held. In June, 1942, at St. Austell, Messrs. F. W. Kellaway, W. G. Tamlin and H. Julian opened a discussion on "The Ideal School Certificate Syllabus", to which everyone present contributed. In February, 1943, at Plymouth, Mr. John Case read a paper on "The Correlation of Science and other branches of Learning", Dr. Margaret Stimson acting as Chairman. In May, 1943, at University College, Exeter, Professor T. Arnold Brown spoke on "Newton—Copernicus—Einstein", the Chairman being Dr. John Murray, Principal of the College.

The attendances at these meetings have been gratifyingly large. Members who are not receiving notices, or who wish to pay subscriptions, are asked to write to the Secretary at H.M. Dockyard School, Montpelier, Plymouth.

F. W. K.

REVIEWS.

Wave Guides. By H. R. L. LAMONT. Pp. vii, 102. 4s. 1942. Monographs on physical subjects. (Methuen)

In the early days of radio, much analytical work was done in investigating the propagation of electromagnetic waves round the earth, *e.g.* H. M. Macdonald's well-known essay. Since then the propagation from aerials of various kinds has been studied intensively. So much work has been published that, superficially, it would appear that nothing of importance has been left undone. But the advance in radio technique, due to the recent development of ultra high frequency (U.H.F.) apparatus ($\lambda < 10$ cm., $f > 3 \cdot 10^9$ cycles per second), has brought about a new situation. Following standard practice, electromagnetic energy can be transmitted from a short-wave generator to an aerial, either by Lecher wires or by a coaxial cable. In either case there is a completely conducting go and return path for the current. At U.H.F. the current is confined mainly to a very thin superficial layer of the conductor and the power loss in the link between generator and aerial may be relatively large. Investigation into the transmission of electromagnetic waves *inside* hollow tubes of uniform cross-section has revealed a way of reducing this power loss. The power is transmitted through the dielectric (air) within the tube as electromagnetic waves, and not in the form of a conducting current. These tubes are called " wave guides ".

The analysis pertaining to the propagation of electromagnetic energy in a wave guide can be divided into two parts, (i) modes of transmission down the tube, (ii) loss due to the tube not being a perfect conductor. The first problem was treated briefly by J. J. Thomson in *Recent Researches* (1893). O. J. Lodge gave an experimental verification at the Royal Institution in 1894. Electromagnetic waves were propagated from " the inside of a hollow cylinder with sparks at the ends of a diameter ; this being a feeble radiator but a very persistent resonator ". Thus Lodge demonstrated, (*a*) transmission of electromagnetic waves from one end of a hollow cylinder, through the air within, to the other ; (*b*) radiation of part of the transmitted energy into space ; (*c*) reflection of energy backwards and forwards between the two ends of the tube which, therefore, functioned as an electromagnetic resonator.

In 1897 Lord Rayleigh treated propagation of electromagnetic waves in hollow perfectly conducting wave guides of infinite length. He showed that two classes of wave were possible. The first is now called a *transverse magnetic* or *E wave*. Its magnetic force vector is in a plane normal to the direction of propagation (transverse) and has no axial component. The second is called a *transverse electric* or *H wave*. Its electric force vector is transverse, and has no longitudinal component. There is an infinity of waves of both types (modes of transmission), as in the case of a rectangular, circular or elliptical membrane.

The monograph under review gives a summary of recent analytical investigations on wave guides and kindred matters. Three main forms of guide are considered, namely, rectangular, circular and elliptical. The mathematical solutions involve circular, Bessel and Mathieu functions. Diagrams are given, based on computations from the formulae, showing the distribution of lines of electric and magnetic force along the guide. There is a " cut-off " frequency below which energy is not transmitted down the guide. This can be explained by reference to geometrical optics, taking group and phase velocity into consideration. A similar effect occurs in a long loud-speaker horn of exponential cross-section.

Transmission within long sectoral and conical horns (wave guides of in-

creasing cross-section) is considered, the analysis involving Hankel functions and tesseral harmonics. Rectangular and cylindrical guides are then considered as electromagnetic resonators, as also are spheres and coaxial closed cylinders, the various vibrational modes being obtained.

Finally, the radiation from open-ended guides and from a sectoral horn is dealt with. The familiar beam or focussing effect due to the horn is illustrated by diagrams, which are similar in character to those obtained with loud-speaker horns and various types of aerial array used for transoceanic short-wave transmission and reception.

The subject of wave guides is one of immediate importance, the technique of which is being developed rapidly for practical purposes. Owing to great advances in the design of thermionic valve transmitters for U.H.F., wave guides have graduated from the laboratory stage of 1894 to one of practical utility. Between 1936 and 1941 inclusive, no fewer than 55 papers, mainly of a mathematical nature, have been published. Unfortunately only two are from this country, one by the author of the monograph under review, the other by one of his colleagues. About 50% of the remainder are from America. This emphasises, even more clearly than ever, the apathy towards applied and technical mathematics which characterises universities, technical colleges, and every institution in Britain where technical work is done. The " hit and miss " principle is still too prevalent, and this defect will not be overcome until the technical faculties of these institutions are populated by staff with the necessary practical experience, and a sound knowledge of higher technical mathematics.*

The wide scope of the subject-matter of the monograph has necessitated a good deal of condensation. But the presentation is logical and clear. It achieves its object, namely, to whet the reader's appetite for more. Any lecturer in applied mathematics who wants to escape from vibrating strings, vibrating bars, and all the toys of a past regime, will find the subject of wave guides much to his liking. It has the additional advantage of being of more than academic interest. All the functions of importance in modern applied and technical mathematics occur in the analysis. As a clear and concise introduction, this monograph can certainly be recommended. N. W. McL.

Mathematics at the Cross-Roads. By A. J. VAN ZYL. Pp. 239. 1942. (Maskew Miller, Ltd., Cape Town)

This is a most refreshing book of interest to anyone who is concerned with reform in the mathematical curriculum in secondary schools. Mr. van Zyl gives an account of the teaching in South Africa, and of the changes he would like to see. But he enters fully into the principles involved and so gives the book an interest to a public wider than that for which it was primarily written. The author studied for a year in England at the University of London Institute of Education, and has travelled very widely, and gives an interesting comparison of the systems in England and America. In addition he gives a very full bibliography.

The school curriculum in South Africa is bound by examinations, which provide matriculation for the universities. Mathematics is not a compulsory subject, and its hitherto undisputed place in the curriculum is now being questioned, thus making the reorganisation of the syllabus most important.

A long survey of the syllabus shows that in some states the arithmetic has been made compulsory and improved by the inclusion of Arithmetic of Citizenship, practical mensuration, and the use of logarithms and formulae.

* Unfortunately, I address the " converted ", but the information can always be passed on to the appropriate quarter.

Algebra contains far too much emphasis on manipulation. Geometry tends to start with a purely logical course and the author, in suggesting improvements, refers at length to the Association's reports on geometry, and makes an interesting statement on the difference between intuitional and experimental geometry.

The present training course for teachers is of one year's duration following a three years' course of mathematics. In this the author finds too little time spent on methods of teaching, and advocates more, spread over two years. I cannot agree with this as I feel that the mathematical teacher needs the spur to general reading on psychology and history, and that additional method teaching is better left for " refresher " courses after practical experience.

The second part of the book is concerned with principles involved in teaching mathematics and with recent trends in other countries. The author considers the utilitarian, cultural and disciplinary values of mathematics, with a good discussion on the possibility of transfer of training of reasoning powers. The third part of the book groups together suggestions for reorganisation.

The production, paper and type are such as to make the book easy and pleasant to read, and without doubt teachers in schools would profit and be stimulated by reading the book. K. S. S.

Map Projections by Practical Construction. By A. HINCKLEY. Pp. 84. 3s. 1942. (Philip)

This book, or rather booklet, contains—just what its title implies plus a few oddments which have some connection with maps, such as Position Circles and Time.

The subject-matter is clear, concise and methodically arranged in the form of notes, well numbered for reference, and each chapter contains an adequate number of exercises, but the material is so condensed that for all except the budding cartographer it should be taken in small doses to avoid indigestion.

The author, in his preface, states that " very little mathematical equipment beyond the ability to draw a plan and elevation " is required but adds that the " work may be linked with Trigonometry and more Advanced Mathematics ", and that, although they are not really necessary, " Four Figure Tables . . . will save labour and, generally, result in greater accuracy ".

In the foreword, Mr. A. R. Hinks, C.B.E., F.R.S., considers that the book " provides ample material for teaching simple geometry in an attractive way ". Except for the " simple " I agree with Mr. Hinks and infer, from the sentence quoted, that he agrees with me in allocating the book to the teacher and not the pupil.

The actual constructions are " simple " enough and are excellent examples of geometrical drawing of the plan-and-elevation variety, but the three-dimensional basis on which they depend is, I am sure, not as simple as all that to the student whose knowledge (by the author's implication) is so elementary that it does not even stretch to the Trigonometrical Ratios.

Mr. Hinckley in attempting to introduce interest value has not succeeded in avoiding the standard pitfalls. For example, he states that " a freely pivoted magnetic needle points approximately N.". His readers would, therefore, unless otherwise informed, be somewhat surprised if they ever found themselves navigating round the north of Canada since at some point during their voyage the north-seeking end of the compass would be pointing due south—which, as a Canadian might say, is *some* approximation for north! Another example, which though theoretically correct is somewhat misleading, occurs where he tells us that the " pin-pointing of position from astronomical

observations depends on two measurements, the Elevation and Bearing of a Star " (incidentally, he really means *three* since the *time* must be known). This wording implies a high degree of accuracy whereas in practical navigation the bearing has to be taken with an ordinary compass and a small error in the bearing may cause an error of many miles in the resulting position : the method is, consequently, of no practical value. G. A. C.

A Source Book of Mathematical Applications. Compiled by a Committee of the National Council of Teachers of Mathematics ; Seventeenth Year Book. Pp. xiii, 291. N.p. 1942. (Bureau of Publications, Teachers College, Columbia University)

" During a recent 80-mile-per-hour windstorm it was found that the top of the Empire State Building in New York swayed through an eight-inch arc at the rate of 7·85 times per minute. At one end of the sway, the building reached $6\frac{3}{4}$ inches out of plumb away from the wind and at the other end it returned $1\frac{1}{4}$ inches beyond plumb toward the wind."

" To have a well-designed chimney it is necessary to take into consideration symmetry of appearance as well as performance of function. If q is the amount of coal used in pounds per hour, and h the height of the chimney, then the area, A, in square feet, of the top of the chimney should be determined by the formula $A = \cdot 03q/\sqrt{h}$. The effective area, E, of the chimney is then given by the formula $E = A - \cdot 6\sqrt{A}$."

" An excellent illustration of the use of polyhedral angles is the revolving winter door at hotels and store entrances."

" How to Calibrate the Sundial " (in full detail).

It would be an unimaginative teacher indeed who could not compile, on bases such as these, sets of examples at once interesting and instructive ; or who failed to enliven his lessons with them. The items quoted are but four of some 700 collected in this Year Book. Quotation is, probably, the best way to indicate the extraordinary scope of the matter dealt with, for although grouped under main headings of Arithmetic, Algebra, Geometry, Trigonometry the subjects included are practically as numerous as the items themselves.

The compilation and arrangement have been done by a central committee appointed by the National Council of Teachers of Mathematics. They do well to point out that applications of the subject are no measure of the significance of mathematics as a whole, and that " to overlook the cultural and aesthetic fruits of the study of mathematics would constitute a grievous error ". At the same time it is certain that the young learner appreciates concrete facts, and is the more eager to learn a subject when he can see its use and its application to the problems of everyday life.

Designed to be of the standard of the mathematics " usually offered in grades seven through twelve " (and roughly equivalent to the work done in our schools in the years leading to the School Certificate examination), the volume will be of inestimable assistance not only to teachers concerned directly with the mathematics of such classes, but also to many others. The first topic quoted above, for example, would have its uses in a study of harmonic motion ; so would the frequent references to musical scales and to sine curves generally. Those dealing with, say, Building or Engineering Science are well catered for ; as are budding photographers, navigators, surveyors, insurance students, biologists. . . . Some of the applications appear merely as statements ; often they are linked to a particular problem, to which, on occasions, the solution is added. In other cases " only the setting for the applications are supplied and the methods of utilization are optional ". A typical possibility, explained in the introduction, comprises a list of eight problems on a

particular item. That number could often be exceeded, and, let it be repeated, there are about 700 separate items.

Some of the subjects are, as it were, " obvious " choices for inclusion (*e.g.* the principle of the micrometer calliper ; the determination of the height of an inaccessible building ; the grains of wheat in G.P. on the squares of a chessboard, and so on), but more often the application is unexpected. How shoe sizes are determined ; the durability of furs (rated relatively to the strongest, which is otter), cosmetology,* the grading of eggs, analyses of popular foodstuffs and medicines—these are topics selected practically at random.

It is clear, then, that plenty of material is available in this book ; it should be made equally clear that it is *useful* material. Hardly ever does one get the impression that a topic has been " dragged in " ; rather does one feel that each is certain to be included in one's next relevant lesson.

Lest this appear fulsome, it may be pointed out that the excellences of the volume have not caused the reviewer to lose his sense of proportion. There are a few flaws apparent ; for instance, the rule for finding the date of Easter should be replaced by that recently publicised by the Astronomer-Royal † ; the index is not perfect (this last-named item cannot be traced by its aid) ; two formulae for indicated horse-power are given, one having the conventional units (PLAN/33,000) while the other becomes Plan$l$/792,000 where l is " length of the piston in square inches " (*sic*) ; and acknowledgements to firms seem unnecessarily lavish. This last is a fault on the right side, perhaps, but it is surprising to find, for example, the standard formula for the deflection of a cantilever used by permission of a particular motor company.

All these are surface blemishes ; the whole remains a most valuable reference-book. The introduction states that " there is no pretence that the list of applications for any topic is exhaustive ; it may not be representative ; there is the hope, however, that it will be suggestive ". This is over-modest. The Year Books of the National Council have invariably been of a high standard ; even so this, the 17th, is above average.

One suggestion the present editorial committee certainly did not have in mind is put forward here : that our own Association prepares a complementary volume to contain as complete a collection of references for work in the somewhat more advanced stages. F. W. K.

The Birthplace and Genesis of Life Assurance and Other Essays. By J. G. ANDERSON. Second edition. Pp. 104. 1940. 5s. (Frederick Muller)

Teachers having intending actuaries among their pupils may well be asked awkward questions : " How, when and where did *scientific* life assurance begin? Who were the pioneers? Of what nature were the early policies and premiums? " Of the seven sections of Mr. Anderson's little book, the longest, " The genesis of life assurance " goes a great way towards answering such questions and it is therefore to be recommended to teachers in search of information. The author, until recently Secretary of the " Equitable ", has, as might be inferred from his note on Bayes in the *Gazette* for July, 1941, a gift for painstaking, scholarly research, and a capacity for relating its results which can make us follow his investigations into the note-books of collectors of sewer-rates with an interest which we usually reserve for the exploits of Mr. Fortune or Dr. Fell. If it is asked what sewer-rates have to do with

* Cosmetology is defined in *Webster* as a treatise on dress and bodily cleanliness. It is the title here given to a paragraph explaining that sinusoidal currents are displacing faradism for the beautician !

† *The Times*, 27th February, 1943, page 5.

life assurance, the answer is that by the note-books of the collectors Mr. Anderson has been able to fix precisely the position of the first life assurance office, that of the "Old Equitable".

Although the long essay mentioned above is perhaps the most interesting part of the book, attention should also be drawn to the final short section "The pioneers of life assurance", where we see old friends in a new light—Halley, De Moivre, Thomas Simpson, Dodson and Mountaine. Mr. Anderson corrects the common belief that Petty and Graunt are to be regarded as the fathers of mortality-statistics by insisting that the first really scientific study was that made by the brilliant and versatile Halley, whose paper in the *Philosophical Transactions* of the Royal Society "may be regarded as the birth of actuarial methods". As Professor A. M. Carr-Saunders * remarks, "The famous astronomer, having promised to present a paper to the Royal Society, which was running short of material, came across data from the city of Breslau which gave not only the facts which Graunt had had for London but in addition the ages of those who had died. Hence he was able to construct the first true life table."

Mr. Anderson tells us in a note that war-time conditions have prevented him from expanding this second edition into a more substantial work; while we are grateful for the present sketches, we may hope that the more complete volume will not go unwritten. T. A. A. B.

* "The growth of the population of Europe", *European Civilisation*, V (Oxford, 1937), p. 360.

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